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Special Functions – 7A

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UNIT-1 Beta- Gamma Functions

Definition: (Beta Function): To define the function Beta by

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$$

And to define other function called Gamma function by

$$\Gamma(n) = \int_0^{\propto} e^{-x}\ x^{n-1}\ dx$$

Properties: 1. Show that $\beta(l,m) = \beta(m,l)$

Solution: $\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$

But $\int_0^a f(x)dx = \int_0^a f(a-x)$ dx

$$\beta(l,m) = \int_0^1 (1-x)^{l-1}[1-(1-x)]^{m-1}dx$$

$$= \int_0^1 (1-x)^{l-1}\ x^{m-1}\ dx$$

$$= \int_0^1 x^{m-1}\,(1-x)^{l-1}\ dx = \beta(m,l)$$

2.Show that $\Gamma(n) = (n-1)\Gamma(n-1)$

Solution: By the definition of Gamma function

$$\Gamma(n) = \int_0^{\propto} e^{-x}\ x^{n-1}\ dx$$

I LATE Formula by Integral by parts $\int u\ v\ dx = u\ v_1 - \int u'\ v_1 dx$ Where $u' = du$ and $v_1 = \int v\ dx$

$$\Gamma(n) = \int_0^{\propto}\ x^{n-1}\ e^{-x}\ dx$$

$$= [\ x^{n-1}(-e^{-x})]_0^{\infty} \ - \ \int_0^{\infty}(n-1)x^{n-1-1}(-e^{-x})dx$$

$$= \ (0-0) + (n-1)\int_0^{\infty} e^{-x}\, x^{(n-1)-1}dx = (n-1)\Gamma(n-1)$$

3. *Show that* $\Gamma(1) = 1$

Solution: By the definition of Gamma function

$$\Gamma(n) = \int_0^{\propto} e^{-x}\ \ x^{n-1}\ \ dx$$

$$\therefore\ \Gamma(1) = \int_0^{\propto} e^{-x}\ \ x^{1-1}\ \ dx = \int_0^{\propto} e^{-x}\ \ \ dx = [\,-e^{-x}]_0^{\infty} = -[e^{-\infty} - e^{-0}] = -\left[\frac{1}{\infty} - 1\right] = 1$$

Note:1. $\Gamma(0) = \infty \quad\ \Gamma(-n) = \infty \quad 2.\ \Gamma(n) = (n-1)\Gamma(n-1)$

3. $\Gamma(n) = (n-1)!$

4.Show that $\int_0^{\infty} e^{-kx}\, x^{n-1}dx = \frac{\Gamma(n)}{k^n}$

Solution: Consider

$\int_0^{\infty} e^{-kx}\, x^{n-1}dx$ (Put $kx = y \Rightarrow x = \frac{y}{k} \ \Rightarrow dx = \frac{1}{k}dy$)

$$=\int_0^{\infty} e^{-y}\, [\tfrac{y}{k}]^{n-1}\tfrac{1}{k}dy = \frac{1}{k^n}\int_0^{\infty} y^{n-1}e^{-y}dy = \frac{1}{k^n}\int_0^{\infty} x^{n-1}e^{-x}dy = \frac{\Gamma(n)}{k^n}$$

5.Show that $\beta(l,m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{l+m}}\ dx = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{l+m}}\ dx$

Solution: By the definition of Beta function

: $\beta(l,m) = \int_0^{1} x^{l-1}(1-x)^{m-1}dx$

Let $x = \frac{1}{y+1}$ $and\ 1 - x = 1 - \frac{1}{y+1} \Rightarrow \frac{y}{y+1}$

$\Rightarrow dx = -\frac{1}{(y+1)^2}dy$

Limits: As $x = \frac{1}{y+1} \ \Rightarrow y + 1 = \frac{1}{x} \ \Rightarrow y = \frac{1}{x} - 1$

Put $x = 0 \Rightarrow y = \infty - 1 = \infty \quad and\ x = 1 \Rightarrow y = \frac{1}{1} - 1 = 0 \quad \therefore\ y = \infty\ to\ 0$

Now $\beta(l,m) = \int_0^{1} x^{l-1}(1-x)^{m-1}dx$

$$= \int_{\infty}^{0}[\tfrac{1}{y+1}]^{l-1}\,[\tfrac{y}{y+1}]^{m-1}[-\tfrac{1}{(y+1)^2}dy]$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(\,y+1)^{l-1+m-1+2}}dy = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{l+m}}dy = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{l+m}}$$

But $\beta(l,m) = \ \beta(m,l)$

Hence $\beta(l,m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{l+m}}\ dx = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{l+m}}\ dx$

Theorem: Relation between Beta and Gamma functions

$$Prove\ that \quad \beta(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$$

Proof: To prove this theorem first note the following identities

1. $\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$ 2 $\Gamma(n) = \int_0^\infty e^{-x}\ x^{n-1}\ dx$

3. $\int_0^\infty e^{-kx}\, x^{n-1}dx = \frac{\Gamma(n)}{k^n}$ and 4. $\beta(l,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}}\ dx$

From (3) $\int_0^\infty e^{-zx}\, x^{l-1}dx = \frac{\Gamma(l)}{z^l} \Rightarrow \int_0^\infty e^{-zx}\, z^l x^{l-1}dx = \Gamma(l)$

Multiply with $e^{-z}z^{m-1}$ *on both sides*

$$\int_0^\infty e^{-zx}\, z^l x^{l-1}\ e^{-z}z^{m-1}dx = \Gamma(l)e^{-z}z^{m-1}$$

$$\int_0^\infty e^{-(1+x)z}\, z^{l+m-1}x^{l-1}dx = \Gamma(l)e^{-z}z^{m-1}$$

$$\Gamma(l)\int_0^\infty e^{-z}z^{m-1}dz = \int_0^\infty [\int_0^\infty e^{-(1+x)z}z^{(l+m)-1}\, dz\ ]x^{l-1}dx$$

By identity number (3)

$$\Gamma(l)\Gamma(m) = \int_0^\infty \frac{\Gamma(l+m)}{(1+x)^{(l+m)}}\ x^{l-1}dx = \Gamma(l+m)\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\ dx$$

$$= \Gamma(l+m)\beta(l,m)$$

Hence $\beta(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$

Note: As $\beta(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$ $put\ l+m=1 \Rightarrow m = 1-l$

$$\beta(l,1-l) = \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(1)} = \Gamma(l)\,\Gamma(1-l) = \frac{\pi}{\sin l\pi}$$

Theorem: Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof: We know that

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$$

Put $x = sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\ d\theta$

And the limits: As $sin^2\theta = x$

Put $x = 0 \Rightarrow sin^2\theta = 0 = sin^2 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow sin^2\theta = 1 = sin^2\pi/2 \Rightarrow \theta = \pi/2$

Now $\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$

$$= \int_0^{\pi/2}(sin^2\theta)^{l-1}(1-sin^2\theta)^{m-1}2\sin\theta\cos\theta\, d\theta$$

$$=2\int_0^{\pi/2} sin^{2l-2}\theta\,(cos^2\theta)^{m-1}\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} sin^{2l-1}\theta\, cos^{2m-1}\theta\, d\theta$$

But $\beta(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$

$$\therefore \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} = 2\int_0^{\pi/2} sin^{2l-1}\theta\, cos^{2m-1}\theta\, d\theta$$

Let $l = m = \frac{1}{2}$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = 2\int_0^{\frac{\pi}{2}} sin^{2\left(\frac{1}{2}\right)-1}\theta\, cos^{2\left(\frac{1}{2}\right)-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} 1 d\theta = 2\,(\theta\,)_0^{\pi/2} = 2\left\{\frac{\pi}{2}\right\} = \pi$$

$$\frac{\Gamma(1/2)^2}{\Gamma(1)} = \pi \quad But\ \Gamma\,(1) = 1$$

$$\therefore\ \Gamma(\tfrac{1}{2})^2 = \pi\ \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Note: $\int_0^{\pi/2} sin^p\,\theta cos^q\theta\, d\theta = \frac{\Gamma[\frac{p+1}{2}]\quad\Gamma[\frac{q+1}{2}]}{2\Gamma[\frac{p+q+2}{2}]}$

Theorem: (Legendre's Duplication Formula)

Statement: Prove that $\Gamma(\mathrm{m})\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\ \Gamma(2\mathrm{m})$

Proof: We know that

$$\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} = 2\int_0^{\pi/2} sin^{2l-1}\theta\, cos^{2m-1}\theta\, d\theta$$

Put $l = \frac{1}{2}$ we get $\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma(\frac{1}{2}+m)} = 2\int_0^{\pi/2} sin^{2\left(\frac{1}{2}\right)-1}\theta\, cos^{2m-1}\theta\, d\theta$

$$\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma(\frac{1}{2}+m)} = 2\int_0^{\pi/2} cos^{2m-1}\theta\, d\theta \quad \text{But } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\frac{\sqrt{\pi}\,\Gamma(m)}{2\,\Gamma(\frac{1}{2}+m)} = \int_0^{\pi/2} cos^{2m-1}\theta\, d\theta \quad \text{----------- (1)}$$

Again

$$\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} = 2\int_0^{\pi/2} sin^{2l-1}\theta\; cos^{2m-1}\theta\; d\theta$$

Put $l = m$

$$\frac{\Gamma(m)\,\Gamma(m)}{\Gamma(m+m)} = 2\int_0^{\pi/2} sin^{2m-1}\theta\; cos^{2m-1}\theta\; d\theta$$

$$\frac{[\,\Gamma(m)\,]^2}{\Gamma(2m)} = \frac{2}{2^{2m-1}}\int_0^{\pi/2} 2^{2m-1} sin^{2m-1}\theta\; cos^{2m-1}\theta\; d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} sin^{2m-1}2\theta\; d\theta$$

Put $2\theta = t \;\Rightarrow \theta = \frac{1}{2}\, t \;\; and \;\; d\theta = \frac{1}{2}\, dt$

$Also\ limits\ Put\ \theta = 0 \;\Rightarrow t = 0 \;\; and \;\; \theta = \pi/2 \;\Rightarrow t = \pi$

$$\frac{[\,\Gamma(m)\,]^2}{\Gamma(2m)} = \frac{1}{2^{2m-2}}\int_0^{\pi} sin^{2m-1}t \quad (\tfrac{1}{2}dt)$$

We know that $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad where\ f(2a-x) = f(x)$

$$\therefore \quad \frac{[\,\Gamma(m)\,]^2}{\Gamma(2m)} = \frac{2}{2^{2m-2}}\int_0^{\pi/2} sin^{2m-1}t \quad (\tfrac{1}{2}dt) = \frac{1}{2^{2m-2}}\int_0^{\pi/2} sin^{2m-1}t\; dt$$

Also $\int_0^{\pi/2} sin^n\theta\; d\theta = \int_0^{\pi/2} cos^n\theta\; d\theta\ (Reduction\ formula)$

$$\therefore \quad \frac{[\,\Gamma(m)\,]^2}{\Gamma(2m)} = \frac{1}{2^{2m-2}}\int_0^{\pi/2} cos^{2m-1}\theta\; d\theta$$

$$\Rightarrow \frac{2^{2m-2}[\,\Gamma(m)\,]^2}{\Gamma(2m)} = \int_0^{\pi/2} cos^{2m-1}\theta\; d\theta \text{ ----------(2)}$$

From (1) and (2)

$$\frac{\sqrt{\pi}\,\Gamma(m)}{2\,\Gamma(\frac{1}{2}+m)} = \frac{2^{2m-2}[\,\Gamma(m)\,]^2}{\Gamma(2m)} \Rightarrow \Gamma(\mathrm{m})\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\;\Gamma(2\mathrm{m})$$

Theorem: Show that $\Gamma\left(-n+\frac{1}{2}\right) = \frac{(-1)^n 2^n\sqrt{\pi}}{1.3.5\ldots\ldots(2n-1)}$

Proof: To prove this theorem first to note the formulae

$$\Gamma(n) = (n-1)\Gamma(n-1)\;\; Where\ n\ is\ positive$$

Replace n by $(n+1)$ $\Gamma(n+1) = (n)\Gamma(n) \;\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \;\; where\ n\ is\ negative$

LHS = $\Gamma\left(-n+\frac{1}{2}\right) = \frac{\Gamma\left(-n+\frac{1}{2}+1\right)}{\left(-n+\frac{1}{2}\right)} = \frac{\Gamma\left(-n+\frac{3}{2}\right)}{\left(-n+\frac{1}{2}\right)}$

$$= \frac{\Gamma\left(-n+\frac{3}{2}+1\right)}{\left(-n+\frac{1}{2}\right)\left(-n+\frac{3}{2}\right)} = \frac{\Gamma\left(-n+\frac{5}{2}\right)}{\left(-n+\frac{1}{2}\right)\left(-n+\frac{3}{2}\right)} \;\;\ldots\ldots\ldots.$$

$$= \frac{\Gamma\left(-n+\frac{2n+1}{2}\right)}{\left(-n+\frac{1}{2}\right)\left(-n+\frac{3}{2}\right)\ldots\ldots\ldots\left(-n+\frac{2n-1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{-2n+2n+1}{2}\right)}{\left(\frac{-2n+1}{2}\right)\left(\frac{-2n+3}{2}\right)\ldots\ldots\ldots\left(\frac{-2n+2n-1}{2}\right)}$$

$$= \frac{2^n\Gamma\left(\frac{1}{2}\right)}{(-2n+1)(-2n+3)\ldots\ldots(-3)(-1)}$$

$$= \frac{2^n\Gamma\left(\frac{1}{2}\right)}{(-1)^n(2n-1)(2n-3)\ldots\ldots3.1} = \frac{(-1)^n 2^n\sqrt{\pi}}{1.3.5\ldots\ldots(2n-1)} = \text{RHS}$$

Theorem: Prove that $2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ldots\ldots(2n-1)\sqrt{\pi}$

Proof: We know that $\Gamma(n) = (n-1)\Gamma(n-1)$ *Where n is positive*

$$\text{LHS} = 2^n\Gamma\left(n+\frac{1}{2}\right) = 2^n\left(n+\frac{1}{2}-1\right)\Gamma\left(n+\frac{1}{2}-1\right)$$

$$= 2^n\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= 2^n\left(n-\frac{1}{2}\right)\left(n-\frac{1}{2}-1\right)\Gamma\left(n-\frac{1}{2}-1\right)$$

$$= 2^n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

= ---

$$= 2^n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\ldots\ldots\left(n-\frac{2n-3}{2}\right)\left(n-\frac{2n-1}{2}\right)\Gamma\left(n-\frac{2n-1}{2}\right)$$

$$= 2^n\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\ldots\ldots\ldots\left(\frac{2n-(2n-3)}{2}\right)\left(\frac{2n-(2n-1)}{2}\right)\Gamma\left(\frac{2n-(2n-1)}{2}\right)$$

$$= \frac{2^n}{2^n}(2n-1)(2n-3)\ldots\ldots\ldots3.1\,\Gamma\left(\frac{1}{2}\right) = 1.3.5\ldots\ldots(2n-1)\sqrt{\pi} = \text{RHS}$$

Problems:

1.Show that $\int_0^1 x^4(1-x)^2dx = \frac{1}{105}$

Solution: We know that

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx\ ,\ \beta(l,m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)} \text{ and } \Gamma(n) = (n-1)!$$

$$\text{LHS} = \int_0^1 x^4(1-x)^2dx = \int_0^1 x^{5-1}(1-x)^{3-1}dx$$

$$= \beta(5,3) = \frac{\Gamma(5)\,\Gamma(3)}{\Gamma(5+3)} = \frac{(5-1)!(3-1)!}{(8-1)!}$$

$$= \frac{4!\,2!}{7!} = \frac{24\times2}{7\times6\times5\times24} = \frac{1}{105} = \text{RHS}$$

2.Show that $\int_0^2 x(8-x^3)^{1/3}\,dx = \frac{16\pi}{9\sqrt{3}}$

Solution:

Let $x^3 = 8t \Rightarrow x = 2t^{1/3}$ $and\ dx = 2.\frac{1}{3}\ t^{-2/3}dt = \frac{2}{3}\ t^{-2/3}dt$

Limits As $t = \frac{1}{8}\ x^3$

Put $x = 0 \Rightarrow t = 0$

Put $x = 2 \Rightarrow t = \frac{1}{8}\ 2^3 = 1$

LHS = $\int_0^2 x(8 - x^3)^{1/3}\, dx = \int_0^1 2t^{\frac{1}{3}}(8 - 8t)^{1/3} \frac{2}{3}\ t^{-2/3}dt$

$$= \frac{4}{3}\int_0^1 8^{\frac{1}{3}}(1-t)^{1/3}\ t^{1/3}\ t^{-2/3}dt$$

$$= \frac{16}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}dt$$

(Let $l - 1 = -\frac{1}{3} \Rightarrow l = \frac{2}{3}$ $and\ m - 1 = \frac{1}{3} \Rightarrow m = \frac{4}{3}$)

$$\int_0^2 x(8-x^3)^{1/3}\, dx = \frac{8}{3}\int_0^1 t^{\frac{2}{3}-1}(1-t)^{\frac{4}{3}-1}dt$$

$$= \frac{8}{3}\beta\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\left(\frac{4}{3}-1\right)\Gamma\left(\frac{4}{3}-1\right)}{\Gamma(2)}$$

$$= \frac{8}{3}\frac{\Gamma\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)}$$

$$= \frac{8}{9}\frac{\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}{(2-1)!} \qquad But\ \Gamma(l)\,\Gamma(1-l) = \frac{\pi}{\sin l\pi}$$

$$= \frac{8}{9.1}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{8}{9}\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{16}{9}\frac{\pi}{\sqrt{3}} = \frac{16\pi}{9\sqrt{3}}$$

3.Evaluate 1. $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$ 2. $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$

Solution:1. $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = \int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}dx$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}dx$$

But $\beta(l,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}}\, dx = \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx$

$$= \beta(9,15) - \beta(15,9) = 0 \qquad (\because\ \beta(l,m) = \beta(m,l))$$

2. $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx = \int_0^\infty \frac{x^4 - x^9}{(1+x)^{15}}dx$

$$= \int_0^{\infty} \frac{x^4}{(1+x)^{15}} dx - \int_0^{\infty} \frac{x^9}{(1+x)^{15}} dx$$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}} dx - \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}} dx$$

But $\beta(l,m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{l+m}}\, dx = \int_0^{\infty} \frac{x^{l-1}}{(1+x)^{l+m}}\, dx$

$$= \beta(5,10) - \beta(10,5)$$

$$= 2\beta(5,10) \quad (\because\ \beta(l,m) = \beta(m,l))$$

$$= 2\,\frac{\Gamma(5)\ \ \Gamma(10)}{\Gamma(5+10)} = 2\ \frac{(5-1)!(10-1)!}{(15-1)!}$$

$$= 2\,\frac{4!\ 9!}{14!} = 2\ \frac{24}{14\times13\times12\times11\times10} = \frac{1}{5005}$$

4.Show that $\int_{-1}^{+1}(1+x)^{p-1}(1-x)^{q-1}\, dx = 2^{p+q-1}\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$

Solution: Let $\frac{1+x}{1-x} = t \Rightarrow 1+x = t(1-x) \Rightarrow x(1+t) = t-1 \Rightarrow x = \frac{t-1}{t+1}$

And $dx = \frac{2}{(t+1)^2} dt$

Limits: As $\frac{1+x}{1-x} = t$ put $x = -1 \Rightarrow t = 0$ $and\ x = +1 \Rightarrow t = \infty$

$1 + x = 1 + \frac{t-1}{t+1} = \frac{2t}{t+1}$

$1 - x = 1 - \frac{t-1}{t+1} = \frac{2}{t+1}$

Now $\int_{-1}^{+1}(1+x)^{p-1}(1-x)^{q-1}\, dx = \int_0^{\infty}[\frac{2t}{t+1}]^{p-1}[\frac{2}{t+1}]^{q-1}\frac{2}{(t+1)^2} dt$

$$= 2^{p-1+q-1+1}\int_0^{\infty}\frac{t^{p-1}}{(t+1)^{p-1+q-1+2}} dt$$

$$= 2^{p+q-1}\int_0^{\infty}\frac{t^{p-1}}{(t+1)^{p+q}} dt$$

$$= 2^{p+q-1}\beta(p,q) = 2^{p+q-1}\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

5.Show that $i)\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi\sec\pi x$

$ii)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Solution: Note that $\Gamma(l)\,\Gamma(1-l) = \frac{\pi}{\sin l\pi}$ $and\ \Gamma(n) = (n-1)\Gamma(n-1)$

$$i)\text{LHS} = \Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)$$

$$= (\tfrac{3}{2}-x-1)\Gamma\left(\tfrac{3}{2}-x-1\right)(\tfrac{3}{2}+x-1)\Gamma\left(\tfrac{3}{2}+x-1\right)$$

$$= (\frac{1}{2} - x)\Gamma\left(\frac{1}{2} - x\right)\ (\frac{1}{2} + x)\Gamma\left(\frac{1}{2} + x\right)$$

$$= \left(\frac{1}{4} - x^2\right)\Gamma\left(\frac{1}{2} - x\right)\Gamma\left(\frac{1}{2} + x\right)$$

$$= \left(\frac{1}{4} - x^2\right)\Gamma\left(\frac{1}{2} - x\right)\Gamma\left(1 - (\frac{1}{2} - x)\right)$$

$$= \left(\frac{1}{4} - x^2\right)\ \frac{\pi}{\sin\left(\frac{1}{2} - x\right)\pi}$$

$$= \left(\frac{1}{4} - x^2\right)\ \frac{\pi}{\sin\left(\frac{\pi}{2} - x\pi\right)} = \left(\frac{1}{4} - x^2\right)\ \frac{\pi}{Cos\ \pi x}$$

$$=\left(\frac{1}{4} - x^2\right)\pi\sec\pi x \quad \text{RHS}$$

$$ii)\Gamma\left(\frac{1}{4}\right)\ \Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\ \Gamma\left(1 - \frac{1}{4}\right) = \frac{\pi}{\sin\pi/4} = \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}$$

6.Evaluate $i)\Gamma\left(-\frac{1}{2}\right)$ $\qquad ii)\ \Gamma\left(-\frac{5}{2}\right)$

Solution: We know that If n is negative number $\Gamma(n) = \frac{\Gamma(n+1)}{n}$

Now $i)\ \Gamma(-1/2\ ) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$ = -2 $\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$

$$ii)\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}} = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{[-\frac{5}{2}][-\frac{3}{2}]} = \frac{\Gamma\left(-\frac{1}{2}\right)}{[-\frac{5}{2}][-\frac{3}{2}]} = \frac{-2\sqrt{\pi}}{\frac{15}{4}} = -\frac{8\sqrt{\pi}}{15}$$

7.Show that $\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

Solution: Consider $\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}}$

Let $x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta}$ $\ and\ d\theta = \frac{1}{2\sqrt{\sin\theta}}\cos\theta\,d\theta$

Limits: As $\sin\theta = x^2$

Put $x = 0 \Rightarrow \sin\theta = 0 = \sin 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \sin\theta = 1 = \sin\pi/2 \Rightarrow \theta = \pi/2$

Now $\int_0^1 \frac{x^2\,dx}{(1-x^4)^{\frac{1}{2}}} = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{(1-sin^2\theta)}} \frac{1}{2\sqrt{\sin\theta}}\cos\theta\,d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\theta}}{\cos\theta}\cos\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\ d\theta = \frac{1}{2}\int_0^{\pi/2} sin^{1/2}\theta\, cos^0\theta\, d\theta$$

But $\int_0^{\pi/2} sin^p\,\theta cos^q\theta\,d\theta = \frac{\Gamma[\frac{p+1}{2}]\ \ \Gamma[\frac{q+1}{2}]}{2\Gamma[\frac{p+q+2}{2}]}$

$$\therefore \int_0^1 \frac{x^2\,dx}{(1-x^4)^{\frac{1}{2}}} = \frac{1}{2}\frac{\Gamma\left[\frac{½+1}{2}\right]\ \Gamma\left[\frac{0+1}{2}\right]}{2\Gamma\left[\frac{½+0+2}{2}\right]}$$

$$= \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\left[\frac{5}{4}-1\right]\Gamma\left(\frac{5}{4}-1\right)}$$

$$= \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{3}{4}\right)}{\left[\frac{1}{4}\right]\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \text{ --------(1)}$$

Consider $\int_0^1 \frac{x^2\,dx}{(1+x^4)^{1/2}}$ Let $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$ $and\ d\theta = \frac{1}{2\sqrt{\tan\theta}}sec^2\theta\, d\theta$

Limits: As $\tan\theta = x^2$

Put $x = 0 \Rightarrow \tan\theta = 0 = \tan 0 \Rightarrow \theta = 0$

$x = 1 \Rightarrow \tan\theta = 1 = \tan\pi/4 \Rightarrow \theta = \pi/4$

Now $\int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{(1+tan^2\theta)}}\frac{1}{2\sqrt{\tan\theta}}sec^2\theta\, d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{1}{\sec\theta}\ \frac{1}{2\sqrt{\tan\theta}}sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}\frac{\sec\theta}{\sqrt{\tan\theta}}d\theta = \frac{1}{2}\int_0^{\pi/4}\frac{1}{\sqrt{\sin\theta\cos\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{1}{\sqrt{2\sin\theta\cos\theta}}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\,d\int_0^{\pi/4}\frac{1}{\sqrt{\sin 2\theta}}\,d\theta$$

Let $2\theta = t \Rightarrow 2d\theta = dt$

$and\ limits\ \theta = 0 \Rightarrow t = 0 \qquad \theta = \pi/4 \Rightarrow t = \pi/2$

$$\int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{1}{2\sqrt{2}}\,d\int_0^{\pi/2}\frac{1}{\sqrt{\sin t}}\,dt = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} sin^{-1/2}t\, cos^0 t\, d\theta$$

But $\int_0^{\pi/2} sin^p\,\theta cos^q\theta\, d\theta = \frac{\Gamma\left[\frac{p+1}{2}\right]\ \Gamma\left[\frac{q+1}{2}\right]}{2\Gamma\left[\frac{p+q+2}{2}\right]}$

$$\therefore \int_0^1 \frac{dx}{(1+x^4)^{\frac{1}{2}}} = \frac{1}{2\sqrt{2}}\frac{\Gamma\left[\frac{-\frac{1}{2}+1}{2}\right]\ \Gamma\left[\frac{0+1}{2}\right]}{2\Gamma\left[\frac{-\frac{1}{2}+0+2}{2}\right]}$$

$$= \frac{1}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4\sqrt{2}}\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \text{ ---------(2)}$$

$$LHS = \int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\frac{1}{4\sqrt{2}}\frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi}{4\sqrt{2}} = \text{RHS}$$

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Special Functions -VII(B)CLP

B. SRINIVASARAO. LECTURER IN MATHEMATICS GDC RVPM

UNIT-IV Legendre's Equation

Definition (Legendre's Differential Equation):

A differential equation is in the form

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

is called Legendre's Differential Equation or Legendre's Equation where n is a constent .

It has two solutions denoted by $P_n(x)$ *and* $Q_n(x)$ and defined by

$$P_n(x)=\frac{1.3.5.....(2n-1)}{n\,!}\left[x^n-\frac{n(n-1)}{2.(2n-1)}\,x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}+----\right]\;and$$

$$Q_n(x)=\frac{n\,!}{1.3.5......(2n+1)}\left[x^{-n-1}-\frac{(n+1)(n+2)}{2.(2n+3)}\,x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+----\right]$$

Theorem: Generating Function for $P_n(x)$

Prove that $P_n(x)$ is the coefficient of h^n in the expansion of the power series of

$$(1-2xh+h^2)^{-1/2}$$

Proof:

Note that $(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+--+\frac{1.3....(2n-3)}{2.4.....(2n-2)}x^{n-1}+\frac{1.3....(2n-1)}{2.4.....(2n)}x^n+\cdots...$

Now $(1-2xh+h^2)^{-1/2}=[1-h(2x-h)]^{-1/2}$

$$=1+\frac{1}{2}h(2x-h)+\frac{1.3}{2.4}\,h^2(2x-h)^2+\cdots+\frac{1.3....(2n-3)}{2.4.....(2n-2)}h^{n-1}(2x-h)^{n-1}$$

$$+\frac{1.3....(2n-1)}{2.4.....(2n)}h^n(2x-h)^n+\ldots....$$

The coefficient of h^n

$$\frac{1.3....(2n-1)}{2.4.....(2n)}(2x)^n+\frac{1.3....(2n-3)}{2.4.....(2n-2)}(n-1)C_1(2x)^{n-2}+\frac{1.3....(2n-5)}{2.4.....(2n-4)}(n-2)C_2(2x)^{n-4}+\ldots......$$

$= \frac{1.3....(2n-1)}{2.4.....(2n)} 2^n x^n + \frac{1.3....(2n-3)(2n-1)2n}{2.4.....(2n-2)2n(2n-1)} (n-1)2^{n-2}x^{n-2}$

$+\frac{1.3....(2n-5)(2n-3)(2n-1)(2n-2)2n}{2.4.....(2n-4)(2n-2)2n(2n-3)(2n-1)} \frac{(n-2)(n-3)}{2!} 2^{n-4}x^{n-4} +.........$

$= \frac{1.3.5.....(2n-1)}{n\,!} \left[x^n - \frac{n(n-1)}{2.(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} + - - - - \right] = P_n(x)$

$\therefore$ The coefficient of h^n in $(1-2xh+h^2)^{-1/2}$ $is\ P_n(x)$

It follows $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

And is called generating function for $P_n(x)$

Theorem: Laplace First definite integration for $P_n(x)$

Prove that $P_n(x) = \frac{1}{\pi}\int_0^{\pi}[\,x \pm \sqrt{x^2-1}\cos\varphi]^n\,d\varphi$

Proof: We know by definite integration

$$\int_0^{\pi} \frac{d\varphi}{a \pm b\cos\varphi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad \text{------------(1)}$$

Put $a = 1 - xh$ and $b = h\sqrt{x^2-1}$

$\therefore\ a^2 - b^2 = (1-xh)^2 - h^2(x^2-1) = 1 - 2xh + h^2$

$$\frac{\pi}{\sqrt{a^2-b^2}} = \pi\,(a^2-b^2)^{-1/2} = \pi(1-2xh+h^2)^{-1/2}$$

$$(1) \Rightarrow\ \pi(1-2xh+h^2)^{-\frac{1}{2}} = \int_0^{\pi} \frac{d\varphi}{a \pm b\cos\varphi}$$

$$= \int_0^{\pi} \frac{d\varphi}{(1-xh) \pm (h\sqrt{x^2-1}\,)\cos\varphi}$$

$$= \int_0^{\pi} \frac{d\varphi}{1-h[x \pm \sqrt{x^2-1}\,)\cos\varphi]}$$

$= \int_0^{\pi}[1 - h\{x\ \pm \sqrt{x^2-1}\,)\cos\varphi\}]^{-1}\,d\varphi$ Let t $=\{x\ \pm \sqrt{x^2-1}\,)\cos\varphi\}$

$$= \int_0^{\pi}[1-ht]^{-1}d\varphi$$

$$\Rightarrow\ \pi\sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi}[1 + ht + (ht)^2 + (ht)^3 + \cdots \dots\dots + (ht)^n + \dots\dots]\,d\varphi$$

Comparing the coefficients of h^n

$$\pi\,P_n(x) = \int_0^{\pi} t^n\ d\varphi$$

$$\Rightarrow\ P_n(x) = \frac{1}{\pi}\int_0^{\pi}[\,x \pm \sqrt{x^2-1}\cos\varphi]^n\,d\varphi$$

Theorem: Laplace Second definite integration for $P_n(x)$

Prove that $P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\varphi}{[\,x\pm\sqrt{x^2-1}\cos\varphi]^{n+1}}$

Proof: We know by definite integration

$$\int_0^{\pi} \frac{d\varphi}{a \pm b\cos\varphi} = \frac{\pi}{\sqrt{a^2-b^2}} \quad \text{------------(1)}$$

Put $a = xh - 1 \ and\ b = h\sqrt{x^2-1}$

$\therefore\ a^2 - b^2 = (xh-1)^2 - h^2(x^2-1) = 1 - 2xh + h^2$

$$\frac{\pi}{\sqrt{a^2-b^2}} = \pi\,(a^2-b^2)^{-1/2} = \pi(1-2xh+h^2)^{-1/2} = \frac{\pi}{h}\left(1-\frac{2x}{h}+\frac{1}{h^2}\right)^{-1/2}$$

$$(1) \Rightarrow \frac{\pi}{h}\left(1-\frac{2x}{h}+\frac{1}{h^2}\right)^{-1/2} = \int_0^{\pi}\frac{d\varphi}{a\pm b\cos\varphi}$$

$$= \int_0^{\pi}\frac{d\varphi}{(xh-1)\pm(h\sqrt{x^2-1}\,)\cos\varphi}$$

$$= \int_0^{\pi}\frac{d\varphi}{h\{\,x\pm\sqrt{x^2-1}\cos\varphi\}-1}$$

Let t $=h\{x \pm \sqrt{x^2-1}\,)\cos\varphi\}$

$$\frac{\pi}{h}\left(1-\frac{2x}{h}+\frac{1}{h^2}\right)^{-1/2} = \int_0^{\pi}\frac{d\varphi}{(t-1)} = \int_0^{\pi}[t-1]^{-1}d\varphi = \int_0^{\pi}\frac{1}{t}\left(1-\frac{1}{t}\right)^{-1}d\varphi$$

$$\Rightarrow \frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}\frac{1}{t}[1+\frac{1}{t}+\frac{1}{t^2}++\cdots\ldots\ldots.+\frac{1}{t^n}+\ldots\ldots.]\,d\varphi$$

$$= \int_0^{\pi}[\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\cdots\ldots\ldots.+\frac{1}{t^{n+1}}+\ldots\ldots.]\,d\varphi$$

$$= \int_0^{\pi}\sum_0^{\infty}\frac{1}{t^{n+1}} = \int_0^{\pi}\sum_0^{\infty}\frac{1}{h^{n+1}\{x\pm\sqrt{x^2-1}\,)\cos\varphi\}^{n+1}}$$

$$\Rightarrow \frac{\pi}{h}\sum_{n=0}^{\infty}\frac{1}{h^n}P_n(x) = \int_0^{\pi}\sum_0^{\infty}\frac{1}{h^{n+1}\{x\pm\sqrt{x^2-1}\,)\cos\varphi\}^{n+1}}$$

$Comparing\ the\ coefficients\ of\ h^{n+1}$

$$\pi\,P_n(x) = \int_0^{\pi}\frac{d\varphi}{[\,x\pm\sqrt{x^2-1}\cos\varphi]^{n+1}}\,d\varphi$$

Orthogonal properties of Legendre's Polynomials

Prove that $i)\ \int_{-1}^{+1}P_m(x)P_n\,(x)dx \qquad where\ m\neq n$

$ii)\ \int_{-1}^{+1}[\,P_n(x)]^2dx = \frac{2}{2n+1} \qquad where\ m = n$

Proof: By the definition of Legendre's Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m(m+1)y = 0$$

And they can be written as

$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n+1)y=0$ and $\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+m(m+1)y=0$

But $y=P_n(x)$ and $y=P_m(x)$ solutions

$$\therefore \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0 \quad \times\ P_m(x)$$

and $\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_m(x)\right]+m(m+1)P_m(x)=0\ \times\ P_n(x)$

$$P_m\ \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n\right]-P_n\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_m\right]+[\,n(n+1)-m(m+1)]P_nP_m=0$$

Integrating on both sides from -1 to +1 wrt x

$$\int_{-1}^{+1}P_m\ \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n\right]dx\ \text{-}\int_{-1}^{+1}P_n\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_m\right]dx$$

$$-[\,n(n+1)-m(m+1)]\int_{-1}^{+1}P_nP_m dx=0$$

$$P_m\left[(1-x^2)\frac{d}{dx}P_n\right]_{-1}^{+1}\ \int_{-1}^{+1}P_m{}'\left[(1-x^2)\frac{d}{dx}P_n\right]dx$$

$$-P_n\left[(1-x^2)\frac{d}{dx}P_m\right]_{-1}^{+1}\ \int_{-1}^{+1}P_n{}'\left[(1-x^2)\frac{d}{dx}P_m\right]dx$$

$$-[\,n(n+1)-m(m+1)]\int_{-1}^{+1}P_nP_m dx=0$$

$$[\,n(n+1)-m(m+1)]\int_{-1}^{+1}P_nP_m dx=0$$

$$\Rightarrow\ \int_{-1}^{+1}P_nP_m dx=0 \qquad for\ m\neq n$$

$$\Rightarrow\ \int_{-1}^{+1}P_n(x)P_m(x)\,dx=0 \qquad for\ m\neq n$$

$ii)$ $\int_{-1}^{+1}[\,P_n(x)]^2dx=\frac{2}{2n+1}$ $\quad where\ m=n$

By the generating function for $P_n(x)$

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^n\,P_n(x)$$

Squaring on both sides $(1-2xh+h^2)^{-1}=\ [\sum_{n=0}^{\infty}h^n\,P_n(x)\,]^2$

sides $(1-2xh+h^2)^{-1}=\sum_0^{\infty}h^{2n}[\,P_n(x)\,]^2+2\sum_{m,n\ \ m\neq n}^{\infty}h^{m+n}\,P_n(x)P_m(x)$

Integrating on both sides from -1 to +1 w r t x

$\int_{-1}^{+1}\frac{1}{1-2xh+h^2}\ dx=\ \sum_0^{\infty}h^{2n}\int_{-1}^{+1}[\,P_n(x)\,]^2dx+0$ $\quad from\ second\ orthogonal\ property$

$$\sum_0^{\infty}h^{2n}\int_{-1}^{+1}[\,P_n(x)\,]^2dx=\ \int_{-1}^{+1}\frac{1}{1-2xh+h^2}\ dx$$

$$=-[\,\frac{1}{2h}\log(1-2xh+h^2)]\ _{-1}^{+1}$$

$$= -\frac{1}{2h} \;[\log(1-2h+h^2) - \log(1+2h+h^2)]$$

$$= -\frac{1}{2h}\log\frac{(1-h)^2}{(1+h)^2} = -\frac{1}{h}[\log(1-h) - \log(1+h)]$$

But $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots.$

And $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots\ldots$

$$\therefore \sum_0^{\infty} h^{2n} \int_{-1}^{+1} [\, P_n(x)\,]^2 dx = -\frac{1}{h}\,[-2h - 2\frac{h^3}{3} - 2\,\frac{h^5}{5} \text{--------} - 2\frac{h^{2n+1}}{2n+1} \text{--}]$$

$$= 2 + 2\,\frac{h^2}{3} + 2\,\frac{h^4}{5} + \cdots \ldots + 2\,\frac{h^{2n}}{2n+1} + ..$$

Comparing the coefficients of h^{2n}

$$\int_{-1}^{+1} [\, P_n(x)\,]^2 dx = \frac{2}{2n+1}$$

Recurrence Formulae

I. Prove that $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Proof: By the generating function

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x)$$

$$Differentiate\ w\ r\ t\ h\ on\ both\ sides$$

$$\frac{-1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(1-2xh+h^2)^{-\frac{3}{2}}(1-2xh+h^2)(x-h) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(x-h)(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x) - 2x\sum_{n=0}^{\infty} nh^n\, P_n(x) + \sum_{n=0}^{\infty} nh^{n+1}\, P_n(x)$$

$$(x-h)\sum_{n=0}^{\infty} h^n\, P_n(x) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x) - 2x\sum_{n=0}^{\infty} nh^n\, P_n(x) + \sum_{n=0}^{\infty} nh^{n+1}\, P_n(x)$$

Comparing coefficient of h^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

II. Prove that $nP_n = xP'_n - P'_{n-1}$

Proof: By the generating function

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n\, P_n(x)$$

$Differentiate\ w\ r\ t\ x\ on\ both\ sides$

$$\frac{-1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2h) = \sum_{n=0}^{\infty} h^n\, P'_n(x)$$

$$h(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} h^n\, P'_n(x)$$

$$(1-2xh+h^2)^{-\frac{3}{2}} = \frac{1}{h}\sum_{n=0}^{\infty} h^n\, P'_n(x) \text{--------(1)}$$

$Again\ Differentiate\ w\ r\ t\ h\ on\ both\ sides$

$$\frac{-1}{2}(1-2xh+h^2)^{-\frac{3}{2}}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(x-h)(1-2xh+h^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(1-2xh+h^2)^{-\frac{3}{2}} = \frac{1}{x-h}\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x) \text{ -------(2)}$$

From (1) and (2)

$$\frac{1}{h}\sum_{n=0}^{\infty} h^n\, P'_n(x) = \frac{1}{x-h}\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

$$(x-h)\sum_{n=0}^{\infty} h^n\, P'_n(x) = h\sum_{n=0}^{\infty} nh^{n-1}\, P_n(x)$$

Comparing coefficient of h^n on both sides

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x)$$

$$\Rightarrow\ nP_n = xP'_n - P'_{n-1}$$

III Prove that $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

Proof: By recurrence formula - I

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Differentiate w r t x on both sides

$$(2n+1)xP'_n + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1} \text{ --------(1)}$$

By Recurrence formula – II $\quad nP_n = xP'_n - P'_{n-1}$

$$xP'_n = nP_n + P'_{n-1} \text{ -------(2)}$$

Put the value (2) in (1)

$$(2n+1)[\, nP_n + P'_{n-1}] + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$$

$$(2n+1)(n+1)P_n = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

$$\Rightarrow (2n+1)P_n = P'_{n+1} - P'_{n-1}$$

IV. Prove that $(n+1)P_n = P'_{n+1} - xP'_n$

Proof: By Recurrence formula -II

$$nP_n = xP'_n - P'_{n-1}$$ (To prove the theorem) ------(1)

By Recurrence formula -III

$$(2n+1)P_n = P'_{n+1} - P'_{n-1}$$ (To prove the theorem)-----(2)

Subtracting (1) and (2)

$$(n+1)P_n = P'_{n+1} - xP'_n$$

V. Prove that $(1-x^2)P'_n = n(P_{n-1} - xP_n)$

Proof: By recurrence formula IV

$$(n+1)P_n = P'_{n+1} - xP'_n$$

Replace n by $(n-1)$

$$nP_{n-1} = P'_n - xP'_{n-1}$$ ---------(1)

By recurrence formula II

$$nP_n = xP'_n - P'_{n-1} \Rightarrow \quad P'_{n-1} = xP'_n - nP_n$$ ------(2)

Put the value in (1)

$$nP_{n-1} = P'_n - x(xP'_n - nP_n)$$

$$nP_{n-1} = P'_n - x^2P'_n + nxP_n$$

$$\Rightarrow \quad (1-x^2)P'_n = n(P_{n-1} - xP_n)$$

VI. Prove that $(1-x^2)P'_n = (n+1)(xP_n - P_{n+1})$

Proof: By recurrence formula - I

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

$$(n+1)xP_n + nxP_n = (n+1)P_{n+1} + nP_{n-1}$$

$$(n+1)[\,xP_n - P_{n+1}] == nP_{n-1} - \ nxP_n$$

$$(n+1)[\,xP_n - P_{n+1}] == n[\,P_{n-1} - \ xP_n]$$ -------(1)

By recurrence formula - V

$$\Rightarrow \quad (1-x^2)P'_n = n(P_{n-1} - xP_n)$$ -------(2)

From (1) and (2)

$$(1-x^2)P'_n = (n+1)[\,xP_n - P_{n+1}]$$

Theorem: State and prove Rodrigue's Formula

Statement: **Prove that**

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof : Let $y=(x^2-1)^n$.

Differentiating, $\frac{dy}{dx}=n(x^2-1)^{n-1}.2x$

$\therefore \quad (x^2-1)\frac{dy}{dx}=2n\,xy.$

Differentiating $(n+1)$ times by Leibnitz Theorem, we have

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+(n+1).\frac{d^{n+1}y}{dx^{n+1}}\,2x-\frac{(n+1)\,n}{2\,!}.\frac{d^ny}{dx^n}\,2$$

$$=2n\left[x.\frac{d^{n+1}y}{dx^{n+1}}+(n+1)\frac{d^ny}{dx^n}.1\right]$$

or $\quad (x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+2x\frac{d^{n+1}y}{dx^{n+1}}-n(n+1)\frac{d^ny}{dx^n}=0$

or $\quad (1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-2x\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\frac{d^ny}{dx^n}=0$

Put $\quad Z=\frac{d^ny}{dx^n}$

$\therefore \quad (1-x^2)\frac{d^2Z}{dx^2}-2x\frac{dZ}{dx}+n(n+1)Z=0$

which is Legendre's equation.

Hence its solution is

$Z=c\,P_n(x)$ where c is a constant

or $\quad \frac{d^ny}{dx^n}=cP_n(x).$ ...(1)

Putting $x=1$, we have

$c=\left(\frac{d^ny}{dx^n}\right)_{x=1}$. Since $P_n(1)=1$

Now $y=(x^2-1)^n=(x+1)^n.(x-1)^n$.

Differentiating n times by Leibnitz's theorem, we have

$$\frac{d^ny}{dx^n}=(x-1)^n.\frac{d^n}{dx^n}(x+1)^n+n.\left\{\frac{d^{n-1}}{dx^{n-1}}(x+1)^n\right\}.n(x-1)^{n-1}+\dots$$
$$+n\left(\frac{d}{dx}(x+1)^n\right)\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+(x+1)^n.\frac{d^n}{dx^n}(x-1)^n$$

$$=(x-1)^n\, n!+n.\frac{n!}{1!}(x+1)\, n\, (x-1)^{n-1}+\ldots$$

$$+n.n(x+1)^{n-1}\frac{n!}{1!}(x-1)+(x+1)^n.n!.$$

Putting $x=1.\left(\frac{d^ny}{dx^n}\right)_{x=1}=(1+1)^n.n!=2^n.n!=c$

$\therefore$ From (1), we have

$$P_n(x)=\frac{1}{c}\frac{d^ny}{dx^n}$$

or
$$P_0(x)=\frac{1}{2^n.n!}\frac{d^n(x^2-1)^n}{dx^n}$$

Problem : 1 *Show that*

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{(3x^2-1)}{2},\ P_3(x)=\tfrac{1}{2}(5x^3-3x)$$

Solution : We know that

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

$$=\{1-h(2x-h)\}^{-1/2}$$

$$=1+\frac{h}{2}(2x-h)+\frac{1.3}{2.4}h^2(2x-h)^2$$

$$+\frac{1.3.5}{2.4.6}h^3(2x-h)^3+\frac{1.3.5.7}{2.4.6.8}h^4(2x-h)^4+\ldots$$

or
$$P_0(x)+h\,P_1(x)+h^2\,P_2(x)+h^3\,P_3(x)+h^4\,P_4(x)+\ldots$$
$$=1+x.h+\tfrac{1}{2}(3x^2-1)\,h^2+\tfrac{1}{2}(5x^3-3x)\,h^3$$
$$+\tfrac{1}{8}(35x^4-30x^2+3)+\ldots$$

Equating the coefficients of like powers of h, we have
$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x)$

Problem-2 *Show that* (i) $P_n(1)=1$
(ii) $P_n(-x)=(-1)^n P_n(x)$.

Hence deduce that $P_n(-1)=(-1)^n$.

Solution : (i) We know that

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

Putting $x=1$

$$\sum_{n=0}^{\infty} h^n P_n(1)=(1-2h+h^2)^{-1/2}$$

$$=(1-h)^{-1}$$

$$=1+h+h^2+\ldots+h^n+\ldots=\sum_{n=0}^{\infty} h^n$$

Equating the coefficient of h^n, we have $P_n(1)=1$,

(ii) We have, $(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty} h^n P_n(x)$

$\therefore$ $(1+2xh+h^2)^{-1/2}=\{1-2x(-h)+(-h)^2\}^{-1/2}$

$$=\sum_{n=0}^{\infty}(-h)^n P_n(x).$$

$$=\sum_{n=0}^{\infty}(-1)^n h^n P_n(x). \qquad \ldots(1)$$

Again $(1+2xh+h^2)^{-1/2}=\{1-2(-x)h+h\}^{2-1/2}$

$$=\sum_{n=0}^{\infty} h^n P_n(-x). \qquad \ldots(2)$$

From (1) and (2), we have

$$\sum_{n=0}^{\infty} h^n P_n(-x\}=\sum_{n=0}^{\infty}(-1)^n h^n P_n(x).$$

Equating the coefficients of h^n,

$$P_n(-x)=(-1)^n P_n(x).$$

Deduction. Putting $x=1$, we have

$$P_n(-1)=(-1)^n P_n(1)$$
$$=(-1)^n. \qquad \text{Since } P_n(1)=1.$$

Problem-3 *Prove that $P_n(0)=0$, for n odd*

and $P_n(0)=\frac{(-1)^{n/2}\, n!}{2^n \{(n/2)!\}^2}$, *for n even.*

Solution: (i) We know that

$$P_n(x)=\frac{1.3.5\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}\ldots\right].$$

When $n=(2m+1)$, odd then

$$P_{2m+1}(x)=\frac{1.3.5\ldots\{2(2m+1)-1\}}{(2m+1)!}\times\left[x^{2m+1}-\frac{(2m+1)(2m+1-1)}{2.\{2(2m+1)-1\}}x^{2m+1-2}+\ldots\right].$$

Putting $x=0$, $P_{2m+1}(0)=0$.

i.e. $P_n(0)=0$, when n is odd.

Also we have

$$\sum_{n=0}^{\infty} h^n P_n(x)=(1-2xh+h^2)^{-1/2}$$

$$\sum_{n=0}^{\infty} h^n P_n(0)=(1+h^2)^{-1/2}=\{1-(-h^2)\}^{-1/2}$$

$$=1+\tfrac{1}{2}.(-h^2)+\frac{1.3}{2.4}(-h^2)^2+\frac{1.3.5}{2.4.6}(-h^2)^3+\ldots+\frac{1.3.5\ldots(2r-1)}{2.4\ldots2r}(-h^2)^r+\ldots$$

Here all powers of h on the R.H.S. are even.
Equating the coefficient of h^{2m} on both sides, we have

$$P_{2m}(0)=\frac{1.3.5\ldots(2m-1)}{2.4.6\ldots2m}(-1)^m=(-1)^m\frac{(2m)!}{2^{2m}(m!)^2}$$

i.e. when $n=2m$, then

$$P_n(0)=\frac{(-1)^{n/2}\, n!}{2^n\{(n/2)!\}^2}.$$

Proved.

Problem-4 $\int_{-1}^{+1}(x^2-1)P_{n+1}P'_n\,dx=\frac{2n(n+1)}{(2n+1)(2n+3)}.$

Solution: From Recurrence formula V, we have

$$(x^2-1)P'_n=n(xP_n-P_{n-1}).$$

$$\therefore \int_{-1}^{+1}(x^2-1)P_{n+1}P'_n\,dx$$

$$=\int_{-1}^{+1} n(xP_n-P_{n-1})P_{n+1}\,dx$$

the other integral being zero since

$$=n\int_{-1}^{+1}xP_nP_{n+1}\,dx, \qquad \int_{-1}^{+1}P_mP_n\,dx=0,\ \text{if } m\neq n$$

$$=n\int_{-1}^{+1}\frac{(n+1)P_{n+1}+nP_{n-1}}{2n+1}P_{n+1}\,dx \text{ from Rec. formula I}$$

$$=\frac{n(n+1)}{2n+1}\int_{-1}^{+1}P^2_{n+1}\,dx+\frac{n^2}{2n+1}\int_{-1}^{+1}P_{n-1}P_{n+1}\,dx$$

$$=\frac{n(n+1)}{(2n+1)}\cdot\frac{2}{2(n+1)+1}+0$$

$$=\frac{2n(n+1)}{(2n+1)(2n+3)}.$$

Problem-5 *Prove that*

(i) $\int_{-1}^{+1} P_n(x)\,dx=0,\ n\neq 0$

and (ii) $\int_{-1}^{+1} P_0(x)\,dx=2.$

Solution: From Rodrigue's formula, we have

$$P_n(x)=\frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore \int_{-1}^{+1} P_n(x)\,dx=\frac{1}{2^n n!}\int_{-1}^{+1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$=\frac{1}{2^n n!}\left\{\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{+1} \quad \ldots(1)$$

Now $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n=\frac{d^{n-1}}{dx^{n-1}}(x+1)^n(x-1)^n$

$$=(x+1)^n\frac{d^{n-1}}{dx^{n-1}}(x-1)^n$$

$$+(n-1).n(x+1)^{n-1}\frac{d^{n-2}}{dx^{n-2}}(x-1)^n+\ldots$$

$$+(x-1)^n\frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$

$$=(x+1)^n\frac{n!}{1!}(x-1)$$

$$+n(n-1)(x+1)^{n-1}\frac{n!}{2!}(x-1)^2+\ldots$$

$$\ldots+(x-1)^n n!(x+1)$$

$=0$ when $x=-1$ or 1

since each term contains $(x-1)$ and $(x+1)$

$\therefore$ from (1), $\int_{-1}^{+1} P_n(x)\,dx=0.$

(ii) We know that $P_0(x)=1.$

$$\therefore \int_{-1}^{+1} P_0(x)\,dx=\int_{-1}^{+1} dx=\left\{x\right\}_{-1}^{+1}=2.$$

Problem -6 *Prove that* (i) $P_n'(1)=\frac{1}{2}n(n+1)$

(ii) $P_n'(-1)=(-1)^{n-1}\frac{1}{2}n(n+1)$.

Solution : $P_n(x)$ satisfies Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)P_n(x)=0$$

$$\therefore\quad (1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0 \qquad \ldots(1)$$

(i) putting $x=1$, in (1), we have

$$-2P_n'(1)+n(n+1)P_n(1)=0$$

$$\therefore\quad P'_n(1)=\tfrac{1}{2}n(n+1),\ \text{since } P_n(1)=1$$

Putting $x=-1$ in (1), we have

$$2P'_n(-1)+n(n+1)P_n(-1)=0$$

or

$$P'_n(-1)=-\tfrac{1}{2}n(n+1)P_n(-1)$$

$$=(-1)^{n-1}.\tfrac{1}{2}n(n+1)\ \text{since } P_n(-1)=(-1)^n$$

Problem - 7 *Prove that*

$$\int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\,dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Solution: From Recurrence formula I, we have

$$(2n+1)xP_n=(n+1)P_{n+1}+nP_{n-1}.$$

Replacing n by $(n-1)$ and $(n+1)$ respectively, we have

$$(2n-1)xP_{n-1}=nP_n+(n-1)P_{n-2}$$

and

$$(2n+3)xP_{n+1}=(n+2)P_{n+2}+(n+1)P_n$$

Multiplying

$$(2n-1)(2n+3)x^2P_{n+1}P_{n-1}=n(n+1)P_n^2+n(n+2)P_nP_{n+2}$$
$$+(n-1)(n+2)P_{n-2}P_{n+2}$$
$$+(n-1)(n+1)P_{n-2}P_n$$

Integrating between the limits -1 to $+1$, we have

$$(2n-1)(2n+3)\int_{-1}^{+1} x^2 P_{n+1}P_{n-1}\,dx=n(n+1)\int_{-1}^{+1}P_n^2\,dx$$

all other integrals being zero

$$=n(n+1)\frac{2}{(2n+1)}$$

$$\therefore\quad \int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\,dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

&&&&&&&

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PAPER 7A. UNIT-3 HERMITE POLYNOMIALS

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Definition (Hermite Differential Equation)

A differential Equation is in the form

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0$$

Where λ *is a constant*,is Hermite Differential Equation

Solution of the Hermite Differential equation is denoted by $\mathrm{H_n(x)}$ and defined by

$$\mathrm{H_n(x)} = \sum_{r=0}^{(\frac{n}{2})} \frac{(-1)^r\, n!}{r\,!\,(n-2r)!}(2x)^{n-2r}$$

Where $\left(\frac{n}{2}\right) = \frac{n}{2}$ $iff\ n\ is\ even\ number$

$= \frac{1}{2}\ (n-1)\ if\ n\ is\ odd\ number$

Theorem: Generating function for $\mathrm{H_n(x)}$

Prove that $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{H_n(x)}$

Proof: Note that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots\ldots$

And $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots + (-1)^n\frac{x^n}{n!} + \ldots\ldots\ldots$

Now $e^{2tx-t^2} = e^{2tx}\ \ e^{-t^2}$

$$= \left[1 + \frac{2tx}{1!} + \frac{(2tx)^2}{2!} + \frac{(2tx)^3}{3!} + \ldots + \frac{(2tx)^n}{n!} + \ldots\right]\left[1 - \frac{t^2}{1!} + \frac{(t^2)^2}{2!} - \ldots + (-1)^n\frac{(t^2)^n}{n!} + \ldots\ldots\ldots\right]$$

$$= \sum_{r=0}^{\infty}\frac{(2tx)^r}{r!}\ \ \sum_{s=0}^{\infty}(-1)^s\ \frac{(t^2)^s}{s!}$$

$$= \sum_{s,r=0}^{\infty}(-1)^s\frac{(2x)^r}{r!\,s\,!}\ t^{r+2s}$$

In the above expansion the coefficient of t^n is

$$\sum_{s=0}^{\infty} \ (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s\,!} \qquad \text{Taking } r+2s=n \Rightarrow r=n-2s$$

As $r = 0 \ to \ \infty \ \Rightarrow r \geq 0 \ \Rightarrow n-2s \geq 0 \Rightarrow n \geq 2s \ \Rightarrow s \leq \frac{n}{2} \ if \ n \ is \ even \ and$

$$s \leq \frac{n-1}{2} \ if \ n \ is \ odd$$

$\therefore$ The coefficient of t^n in e^{2tx-t^2}

$$\sum_{s=0}^{(\frac{n}{2})} \ (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!\, s\,!} \quad \text{where } \left(\frac{n}{2}\right) = \frac{n}{2} \ iff \ n \ is \ even \ number$$

$$= \frac{1}{2}\,(n-1) \ if \ n \ is \ odd \ number$$

The coefficient of t^n in e^{2tx-t^2}

$$\frac{1}{n\,!}\sum_{s=0}^{(\frac{n}{2})} \ (-1)^s \frac{n! \ (2x)^{n-2s}}{(n-2s)!\, s\,!} \quad \text{where } \left(\frac{n}{2}\right) = \frac{n}{2} \ iff \ n \ is \ even \ number$$

$$= \frac{1}{2}\,(n-1) \ if \ n \ is \ odd \ number$$

The coefficient of t^n in e^{2tx-t^2} is $\frac{1}{n\,!}\mathrm{H_n}(\mathrm{x})$

It follows $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n}(\mathrm{x})$

Theorem (Other form of $\mathrm{H_n}(\mathrm{x})$)

Prove that

$$\mathrm{H_n}(\mathrm{x}) = \ 2^n \left\{ \exp(-\frac{1}{4}\frac{d^2}{dx^2}) x^n \right\}$$

Proof: We know that $\frac{d}{dx}(e^{2tx}) = 2te^{2tx}$

$$\frac{d^2}{dx^2}(e^{2tx}) = (2t)^2 e^{2tx}$$

In general, $\frac{d^n}{dx^n}(e^{2tx}) = (2t)^n e^{2tx}$

$$\Rightarrow \frac{1}{2^n}\frac{d^n}{dx^n}(e^{2tx}) = t^n e^{2tx}$$

$$\Rightarrow \ \left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx} = t^n e^{2tx} \text{ -------(1)}$$

Now $\left\{ \exp(-\frac{1}{4}\frac{d^2}{dx^2}) e^{2tx} \right\} = [\sum_{n=0}^{\infty} \frac{1}{n\,!} \left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n] \, e^{2tx}$

$$= \ \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} \left(\frac{1}{2} \ \frac{d}{dx}\right)^{2n} e^{2tx} \text{ but from (1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} t^{2n} e^{2tx}$$

$$= e^{2tx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n\,!} t^{2n}$$

$$= e^{2tx} \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n\,!} = \qquad =$$

$$= e^{2tx} e^{-t^2} = e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)}$$

$$\therefore \left\{\exp(-\frac{1}{4}\frac{d^2}{dx^2}) e^{2tx}\right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)}$$

$$\left\{\exp(-\frac{1}{4}\frac{d^2}{dx^2})\right\} \sum_{n=0}^{\infty} \frac{(2tx)^n}{n\,!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)}$$

Comparing the coefficients of t^n

$$\left\{\exp(-\frac{1}{4}\frac{d^2}{dx^2})\right\} \frac{(2x)^n}{n\,!} = \frac{1}{n!} \mathrm{H_n(x)}$$

$$\Rightarrow \quad \mathrm{H_n(x)} = 2^n \left\{\exp(-\frac{1}{4}\frac{d^2}{dx^2}) x^n\right\}$$

Rodrigues Formula

Prove that $\mathrm{H_n(x)} = (-1)^n e^{x^2} \frac{d^n}{dx^n} \; e^{-x^2}$

Proof: By the generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)} \quad \text{~~~~~~~~~~(1)}$$

But $2tx - t^2 = -(t^2 - 2tx) = -(t^2 - 2tx + x^2 - x^2) = -(t-x)^2 + x^2 = x^2 - (t-x)^2$

$$(1) \Rightarrow \; e^{x^2-(t-x)^2}$$

$$= \frac{t^0}{0\,!} H_0(x) + \frac{t^1}{1!} H_1(x) + \frac{t^2}{2!} H_2(x) \; + \text{~~~~} + \frac{t^n}{n!} H_n(x) + \frac{t^{n+1}}{(n+1)!} H_{n+1}(x) + \text{~~~~~}$$

Differentiate partially w r t 't' with n times and then put $t = 0$

$$\frac{\partial^n}{\partial t^n} e^{x^2-(t-x)^2} = \frac{n!}{n!} H_n(x)$$

$$H_n(x) = e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2}$$

Put $t - x = u$ *but* $t = 0 \quad \Rightarrow -x = u$ *and* $\frac{\partial^n}{\partial u^n} = \frac{\partial^n}{\partial (-x)^n} = (-1)^n \frac{\partial^n}{\partial x^n}$

$$H_n(x) = e^{x^2} \frac{\partial^n}{\partial t^n} e^{-(t-x)^2} = e^{x^2} \frac{\partial^n}{\partial u^n} e^{-u^2} = e^{x^2} (-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2}$$

Hence $\quad \mathrm{H_n(x)} = (-1)^n e^{x^2} \frac{d^n}{dx^n} \; e^{-x^2}$

Theorem: (Orthogonal Properties of $\mathrm{H_n(x)}$)

Prove that $\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{H_n(x)H_m(x)}dx = 0 \quad if\ m \neq n, \qquad = \sqrt{\pi}\ 2^n\ n\,! \quad if\ m = n$

Proof: By the generating function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{H_n(x)}$$

And $$e^{2sx-s^2} = \sum_{m=0}^{\infty} \frac{s^m}{m!}\mathrm{H_m(x)}$$

Multiply the above

$$e^{2tx-t^2}e^{2sx-s^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathrm{H_n(x)} \quad \sum_{m=0}^{\infty} \frac{s^m}{m!}\mathrm{H_m(x)}$$

$$\sum_{n,m=0}^{\infty} \frac{t^n s^m}{n!m!}\mathrm{H_n(x)H_m(x)} = e^{2tx-t^2}e^{2sx-s^2}$$

$$\frac{1}{n!m!}\mathrm{H_n(x)H_m(x)} = The\ coeff\ of\ t^n s^m in\ e^{2tx-t^2}e^{2sx-s^2}$$

Multiply with e^{-x^2} *and apply integration w r t x from* $-\infty$ *to* $+\infty$

$$\int_{-\infty}^{+\infty} e^{-x^2}\frac{1}{n!m!}\mathrm{H_n(x)H_m(x)}dx = The\ coeff\ of\ \ t^n s^m in \int_{-\infty}^{+\infty} e^{-x^2}e^{2tx-t^2}e^{2sx-s^2}dx \quad \text{---(1)}$$

Consider

$$\int_{-\infty}^{+\infty} e^{-x^2}e^{2tx-t^2}e^{2sx-s^2}dx = \int_{-\infty}^{+\infty} e^{-(x^2+t^2+s^2-2tx-2sx)}dx \quad \text{---------(2)}$$

But $(x-t-s)^2 = x^2 + t^2 + s^2 - 2xt + 2st - 2xs$

$$\therefore -(x^2+t^2+\ s^2-2tx-2sx)$$

$$= -(x^2+t^2+s^2-2xt-2xs+2st-2st)$$

$$= -(x-t-s)^2 + 2st$$

$$\therefore\ e^{-(x^2+t^2+s^2-2tx-2sx)} = e^{-(x-t-s)^2}e^{2st}$$

From(2) $\int_{-\infty}^{+\infty} e^{-x^2}e^{2tx-t^2}e^{2sx-s^2}dx = e^{2st}\int_{-\infty}^{+\infty} e^{-(x-t-s)^2}dx$

$$= e^{2st}\int_{-\infty}^{+\infty} e^{-u^2}dx \quad where\ u = x-t-s$$

$$== e^{2st}\sqrt{\pi}$$

$$= \sqrt{\pi}[\ 1 + \frac{2st}{1!} + \frac{(2st)^2}{2!} + ---- + \frac{(2st)^n}{n!} + ---$$

From (1) $\int_{-\infty}^{+\infty} e^{-x^2}\frac{1}{n!m!}\mathrm{H_n(x)H_m(x)}dx = \sqrt{\pi}\ \frac{2^n}{n!} \qquad for\ n = m$

=0 for n$\neq m$

$$\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{H_n(x)H_m(x)}dx = 0 \quad if\ m \neq n, \qquad = \sqrt{\pi}\ 2^n\ n\,! \quad if\ m = n$$

Recurrence Formulae:

I. Prove that $H'_n(x) = 2nH_{n-1}(x)$

Proof: By the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)} = e^{2tx-t^2}$$

Differentiate w r t x on both sides

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H'_n(x)} = 2t \ e^{2tx-t^2}$$

Again, by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H'_n(x)} = 2t \ \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H'_n(x)} = 2 \ \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} \mathrm{H_n(x)}$$

Now comparing the coefficients of t^n

$$\frac{\mathrm{H'_n(x)}}{n\,!} = 2\frac{1}{(n-1)!} \mathrm{H_{n-1}(x)} \Rightarrow H'_n(x) = 2nH_{n-1}(x)$$

II. Prove that $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

Proof: By the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)} = e^{2tx-t^2}$$

Differentiate w r t t on both sides

$$\sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} \mathrm{H_n(x)} = (2x-2t)e^{2tx-t^2}$$

$$\sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} \mathrm{H_n(x)} = (2x-2t) \ \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)}$$

$$\sum_{n=0}^{\infty} \frac{t^{n-1}}{(n-1)!} \mathrm{H_n(x)} = 2x \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)} - 2 \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} \mathrm{H_n(x)}$$

Now comparing the coefficients of t^n

$$\frac{1}{n!} \mathrm{H_{n+1}(x)} = 2\mathrm{x} \ \frac{1}{n!} \mathrm{H_n(x)} - 2 \ \frac{1}{(n-1)!} \mathrm{H_{n-1}(x)}$$

$$\frac{1}{n!} \mathrm{H_{n+1}(x)} = 2\mathrm{x} \ \frac{1}{n!} \mathrm{H_n(x)} - 2 \ \frac{n}{n!} \mathrm{H_{n-1}(x)}$$

$$\Rightarrow 2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$$

III. Prove that $H'_n(x) = 2xH_n(x) - H_{n+1}(x)$

Proof: By recurrence formula -I

$$H'_n(x) = 2nH_{n-1}(x) \text{ ------(1)}$$

Also, by recurrence formula -II

$$2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x) \text{ -----(2)}$$

(1) – (2) $\Rightarrow H'_n(x) = 2xH_n(x) - H_{n+1}(x)$

IV. Prove that $H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$

Proof: By the Hermite Differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 \text{ But } H_n(x) \text{ is the solution of the D.E}$$

$$\therefore\ H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

Problems:

1.Show that $H''_n(x) = 4n(n-1)H_{n-2}(x)$

Solution: By recurrence formula-I

$$H'_n(x) = 2nH_{n-1}(x) \text{ -------(1)}$$

Differentiate w r t x

$$H''_n(x) = 2nH'_{n-1}(x) \text{ ------(2)}$$

Again from (1) $H'_n(x) = 2nH_{n-1}(x)$

Replace n by (n-1)

$$H'_{n-1}(x) = 2(n-1)H_{n-2}(x) \quad \text{put the value in (2)}$$

$$H''_n(x) = 4n(n-1)H_{n-2}(x)$$

2.Prove that $i)H_{2n}(0) = (-1)^n \frac{(2n)!}{n\,!}$ $\quad ii)H_{2n+1}(0) = 0$

Solution: i) By the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(x)} = e^{2tx-t^2}$$

Put $x = 0$ $\qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(0)} = e^{-t^2}$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(0)} = 1 - \frac{t^2}{1\,!} + \frac{(t^2)^2}{2!} - \text{---------} + (-1)^n \frac{(t^2)^n}{n\,!} + \text{---------}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{H_n(0)} = 1 - \frac{t^2}{1\,!} + \frac{t^4}{2!} - \text{---------} + (-1)^n \frac{t^{2n}}{n\,!} + \text{---------}$$

Comparing the coefficients of t^{2n}

$$\frac{1}{(2n)!}\mathrm{H_{2n}(0)} = (-1)^n \frac{1}{n\,!} \Rightarrow H_{2n}(0) = (-1)^n \frac{(2n)!}{n\,!}$$

ii) Again, to comparing the coefficients of t^{2n+1}

$$\frac{1}{(2n+1)!}\mathrm{H_{2n+1}(0)} = 0 \ \ (\textit{Since there is no odd power t in the RHS})$$

3.Show that $\frac{d^m}{dx^m}$ [$H_n(x)$] $= \frac{2^m\, n!}{(n-m)!} H_{n-m}(x)$

Solution: By the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2tx-t^2}$$

Apply $\frac{d^m}{dx^m}$ on both sides

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n(x) = \frac{d^m}{dx^m} e^{2tx-t^2} = (2t)^m e^{2tx-t^2}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n(x) = (2t)^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

$$= 2^m \sum_{n=0}^{\infty} \frac{t^{n+m}}{n!} H_n(x)$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H_n(x) = 2^m \sum_{n=0}^{\infty} \frac{t^{n+m}}{n!} H_n(x)$$

Put $r = n + m \Rightarrow n = r - m \quad for\, n = 0 \Rightarrow r = m \;\; and\, for\, n = \infty \Rightarrow r = \infty$

Comparing the coefficient of t^n

$$\frac{1}{n\,!} \frac{d^m}{dx^m} H_n(x) = 2^m \frac{1}{(n-m)!} H_{n-m}(x)$$

Hence $\frac{d^m}{dx^m}$ [$H_n(x)$] $= \frac{2^m\, n!}{(n-m)!} H_{n-m}(x)$

All the best

Chapter-3 Bessel's Polynomial

Definition: A differential equation is in the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

Is called Bessel's Differential equation.

And it having two solutions Denoted by $J_n(x)\ and\ J_{-n}(x)$ and defined by

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \text{ and}$$

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Note that

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\ldots\ldots$$

Recurrence formula – I

Prove that $xJ'_n(x)=n\ J_n(x)-x\,J_{n+1}(x)$

Proof: We know that

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating $wrt\ x$

$$J'_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{2}$$

$$=\frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=\frac{n}{x}\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}+\frac{1}{x}\sum_{r=0}^{\infty}\frac{(-1)^r(2r)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$=\frac{n}{x}\ J_n(x)+\frac{2}{x}\sum_{r=0}^{\infty}\frac{(-1)^r}{(r-1)\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\qquad \text{Put } r-1=s\Rightarrow r=s+1$$

$$J'_n(x) = \frac{n}{x}\ J_n(x) + \frac{2}{x}\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{s\,!\,\Gamma(n+1+s+1)}\left(\frac{x}{2}\right)^{n+2(s+1)}$$

$$= \frac{n}{x}\ J_n(x) - \frac{2}{x}\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s\,!\,\Gamma[(n+1)+s+1]}\left(\frac{x}{2}\right)^{n+1+2s+1}$$

$$= \frac{n}{x}\ J_n(x) - \frac{2}{x}\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s\,!\,\Gamma[(n+1)+s+1]}\left(\frac{x}{2}\right)^{n+1+2s}.\left(\frac{x}{2}\right)$$

$$x\,J'_n(x) \ = n\ J_n(x) - x\,J_{n+1}(x)$$

Recurrence formula – II

Prove that $xJ'_n(x) = -n\ J_n(x) + x\,J_{n-1}(x)$

Proof: We know that

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$Differentiating\ wrt\ x$$

$$J'_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}(n+2r)\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{2}$$

$$xJ'_n(x) \ = \ \sum_{r=0}^{\infty}\frac{(-1)^r(2n+2r-n)}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= -n\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty}\frac{(-1)^r(2n+2r)}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= -n\ J_n(x) + \sum_{r=0}^{\infty}\frac{(-1)^r 2(n+r)}{(r-1)!\ (n+r)\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1}\left(\frac{x}{2}\right) \qquad \because \Gamma(n+1) = n\Gamma(n)$$

$$x\,J'_n(x) = -n\ J_n(x) + x\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$= -n\ J_n(x) + x\,J_{n-1}(x)$$

Hence $xJ'_n(x) = -n\ J_n(x) + x\,J_{n-1}(x)$

Recurrence formula – III

Prove that $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

Proof: We know that

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$Differentiating\ wrt\ x$$

$$2\,J'_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} (n+2r) \left(\frac{x}{2}\right)^{n+2r-1}$$

$$2J'_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+r+r)}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

But $\Gamma(n+1) = n\Gamma(n)$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} + \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)}{r!\,(n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

Put $r-1 = s \Rightarrow r = s+1$

$$2\,J'_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^{s+1}}{s\,!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$= -\sum_{r=0}^{\infty} \frac{(-1)^s}{s\,!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s} + J_{n-1}(x)$$

Hence $2\,J'_n(x) = -\,J_{n+1}(x) + J_{n-1}(x)$

Recurrence formula – IV

Prove that $2n\,J_n(x) = x[\,J_{n-1}(x) + J_{n+1}(x)]$

Proof: We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\therefore\ 2n\,J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2n}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\,[\,2n+2r-2r]}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -2\sum_{r=0}^{\infty} \frac{(-1)^r\,(r)}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= -2\sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right) + \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r)}{r!\,(n+r)\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} \left(\frac{x}{2}\right)$$

Put $r-1 = s \Rightarrow r = s+1$

$$= -x\sum_{r=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+2(s+1)-1} + x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$= x\sum_{r=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s} + xJ_{n-1}(x)$$

Hence $2n\, J_n(x) = x[\, J_{n-1}(x) + J_{n+1}(x)]$

Recurrence formula – V

Prove that $\frac{d}{dx}\,[\, x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Proof: L H S = $\frac{d}{dx}\,[\, x^{-n} J_n(x)]$

$$= [-n x^{-n-1} J_n(x)] + x^{-n}\, J'_n(x)$$

$$= [-n x^{-n-1} J_n(x)] + x^{-n-1} x^1\, J'_n(x)$$

$$= x^{-n-1}[-n J_n(x) + x\, J'_n(x)]$$

But by Recurrence formula – I $\quad xJ'_n(x) = n\; J_n(x) - x\, J_{n+1}(x)$

$$\therefore\ \frac{d}{dx}\,[\, x^{-n} J_n(x)] = x^{-n-1}[-n J_n(x) + \; n\; J_n(x) - x\, J_{n+1}(x)]$$

$$= x^{-n-1}[\, -x\, J_{n+1}(x)\,] = = -x^{-n} J_{n+1}(x) = RHS$$

Recurrence formula – VI

Prove that $\frac{d}{dx}\,[\, x^{n} J_n(x)] = x^{n} J_{n-1}(x)$

Proof: L H S = $\frac{d}{dx}\,[\, x^{n} J_n(x)]$

$$= [n x^{n-1} J_n(x)] + x^{n}\, J'_n(x)$$

$$= [n x^{n-1} J_n(x)] + x^{n-1} x^1\, J'_n(x)$$

$$= x^{n-1}[n J_n(x) + x\, J'_n(x)]$$

But by Recurrence formula – II $\quad xJ'_n(x) = -n\; J_n(x) + x\, J_{n-1}(x)$

$$\therefore\ \frac{d}{dx}\,[\, x^{n} J_n(x)] = x^{n-1}[n J_n(x) - \; n\; J_n(x) + x\, J_{n-1}(x)]$$

$$= x^{n-1}[\, x\, J_{n-1}(x)\,] = = x^{n} J_{n-1}(x) = RHS$$

Theorem: ***Prove that if n is the positive integer the coefficient of z^n in the expansion of $e^{x\left(z-\frac{1}{z}\right)/2}$ in ascending and decending powers of z. Also prove that $J_n(x)$ is the coefficient of z^{-n} multiplied by $(-1)^n$ in the expansion of above experssion.***

Proof: $e^{x\left(z-\frac{1}{z}\right)/2}=e^{xz/2}.e^{-x/2z}$

$$=\left[1+\frac{xz}{2}+\frac{1}{2!}\left(\frac{xz}{2}\right)^2+\ldots+\frac{1}{n!}\left(\frac{xz}{2}\right)^n+\frac{1}{(n+1)!}\left(\frac{xz}{2}\right)^{n+1}+\ldots+\frac{1}{(n+2)!}\left(\frac{xz}{2}\right)^{n+2}+\ldots\right].\left[1-\frac{x}{2z}+\frac{1}{2!}\left(\frac{x}{2z}\right)^2+\ldots+\frac{(-1)^n}{n!}\left(\frac{x}{2z}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2z}\right)^{n+1}\right.$$

Coefficient of z^n in this product

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)^{n+1}+\frac{1}{(n+2)!}\,\frac{1}{2!}\cdot\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}+\ldots$$

$$=\frac{1}{n!}\left(\frac{x}{2}\right)^n-\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{2!\,(n+2)!}\cdot\left(\frac{x}{2}\right)^{n+4}+\ldots$$

$$=\frac{(-1)^0}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n+\frac{(-1)}{1!\,\Gamma(n+2)}\cdot\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^2}{2!\cdot\Gamma(n+3)}\cdot\left(\frac{x}{2}\right)^{n+4}+\ldots$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\cdot\Gamma(n+r+1)}\cdot\left(\frac{x}{2}\right)^{n+2r}$$

$$=J_n(x).$$

Similarly, the coefficient of z^{-n} in the product

$$=\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n+\frac{(-1)^{n+1}}{(n+1)!}\,\frac{x}{2}\cdot\left(\frac{x}{2}\right)^{n+1}+\frac{(-1)^{n+2}}{(n+2)!}\frac{1}{2!}\left(\frac{x}{2}\right)^2\left(\frac{x}{2}\right)^{n+2}+\ldots$$

$$=(-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n+\frac{(-1)}{\Gamma(n+2)}\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^2}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^{n+4}+\text{-------------}\right]$$

$$=(-1)^nJ_n(x)$$

Problems:

1. *Prove that*

$i) J_{-n}(x) = (-1)^n J_n(x)$

$ii) J_n(-x) = (-1)^n J_n(x)$ **For x is positive.**

Solution: $i)$**By the definition of** $J_{-n}(x)$

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r} \quad \text{Let } r = n+s$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(n+s)\,!\,\Gamma(-n+(n+s)+1)} \left(\frac{x}{2}\right)^{-n+2(n+s)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{(n+s)\,!\,\Gamma(s+1)} \left(\frac{x}{2}\right)^{n+2s)}$$

$$= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s\,!\,\Gamma(n+s+1)} \left(\frac{x}{2}\right)^{n+2s} = (-1)^n J_n(x)$$

$ii)$***We know that*** $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

If n is positive and to replace $x\ by - x$

$$J_n(-x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{-x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(-1)^{n+2r}}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n(x) \qquad (\,Sincse\ (-1)^{2r} = +1)$$

2.Show that $i) J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ $\qquad ii)\ J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\ \text{Sin}\ x$

Solution: i) We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{(-1)^0}{0\,!\,\Gamma(n+0+1)} \left(\frac{x}{2}\right)^{n+2(0)} + \frac{(-1)^1}{1\,!\,\Gamma(n+1+1)} \left(\frac{x}{2}\right)^{n+2(1)} + \frac{(-1)^2}{2\,!\,\Gamma(n+2+1)} \left(\frac{x}{2}\right)^{n+2(2)} + \text{---------------}$$

$$= \frac{1}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,(n+1)\,\Gamma(n+1)} \left(\frac{x}{2}\right)^n \left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{2\,!(n+2)(n+1)\Gamma(n+1)} \left(\frac{x}{2}\right)^n \left(\frac{x}{2}\right)^4 + \text{---------------}$$

$$= \frac{1}{\Gamma(n+1)} \left(\frac{x}{2}\right)^n \left[\,1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} + \text{-------------}\right]$$

Put $n = -\frac{1}{2}$

$$J_{-1/2}(x) = \frac{1}{\Gamma(-\frac{1}{2}+1)} \left(\frac{x}{2}\right)^{-1/2} \left[\,1 - \frac{x^2}{4\left(-\frac{1}{2}+1\right)} + \frac{x^4}{32\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)} + \text{-------------}\right]$$

$= \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{-1/2}[1 - \frac{x^2}{4\left(\frac{1}{2}\right)} + \frac{x^4}{32\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)} + \text{-------------}]$

$= \frac{1}{\sqrt{\pi}}\left(\frac{2}{x}\right)^{1/2}[1 - \frac{x^2}{2} + \frac{x^4}{24} + \text{-------------}]$

$= \sqrt{\frac{2}{\pi x}}[1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} + \text{-------------}] = \sqrt{\frac{2}{\pi x}}\cos x$

i) Again

$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$= \frac{(-1)^0}{0\,!\,\Gamma(n+0+1)}\left(\frac{x}{2}\right)^{n+2(0)} + \frac{(-1)^1}{1\,!\,\Gamma(n+1+1)}\left(\frac{x}{2}\right)^{n+2(1)} + \frac{(-1)^2}{2\,!\,\Gamma(n+2+1)}\left(\frac{x}{2}\right)^{n+2(2)} + \text{---------------}$

$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,(n+1)\,\Gamma(n+1)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{2\,!(n+2)(n+1)\Gamma(n+1)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^4 + \text{---------------}$

$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} + \text{-------------}]$

Put $n = \frac{1}{2}$

$J_{1/2}(x) = \frac{1}{\Gamma(\frac{1}{2}+1)}\left(\frac{x}{2}\right)^{1/2}[1 - \frac{x^2}{4\left(\frac{1}{2}+1\right)} + \frac{x^4}{32\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} + \text{-------------}]$

$= \frac{1}{\frac{1}{2}\,\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{1/2}[1 - \frac{x^2}{4\left(\frac{3}{2}\right)} + \frac{x^4}{32\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)} + \text{-------------}]$

$= \frac{2}{\sqrt{\pi}}\sqrt{\left(\frac{x}{2}\right)}[1 - \frac{x^2}{6} + \frac{x^4}{120} + \text{-------------}]$

$= \frac{2}{\sqrt{\pi}}\sqrt{\left(\frac{x}{2}\right)}\frac{1}{x}[x - \frac{x^3}{6} + \frac{x^5}{120} + \text{-------------}]$

$= \sqrt{\frac{2}{\pi x}}[x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \text{-------------}] = \sqrt{\frac{2}{\pi x}}\,\text{Sin}\,x.$

3.Show that $i)\sqrt{\frac{\pi x}{2}}\;\; J_{\frac{3}{2}}(x) = \frac{1}{x}\sin x - \cos x$

$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$= \frac{(-1)^0}{0\,!\,\Gamma(n+0+1)}\left(\frac{x}{2}\right)^{n+2(0)} + \frac{(-1)^1}{1\,!\,\Gamma(n+1+1)}\left(\frac{x}{2}\right)^{n+2(1)} + \frac{(-1)^2}{2\,!\,\Gamma(n+2+1)}\left(\frac{x}{2}\right)^{n+2(2)} + \text{---------------}$

$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!\,(n+1)\,\Gamma(n+1)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^2 + \frac{(-1)^2}{2\,!(n+2)(n+1)\Gamma(n+1)}\left(\frac{x}{2}\right)^n\left(\frac{x}{2}\right)^4 + \text{---------------}$

$= \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} + \text{-------------}]$

$Put\ n\ = \frac{3}{2}$

$$J_{3/2}(x) = \frac{1}{\Gamma(\frac{3}{2}+1)}\left(\frac{x}{2}\right)^{3/2}\left[1 - \frac{x^2}{4\left(\frac{3}{2}+1\right)} + \frac{x^4}{32\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)} + \text{----------------}\right]$$

$$= \frac{x\sqrt{x}}{2\sqrt{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}\left[1 - \frac{x^2}{4\left(\frac{5}{2}\right)} + \frac{x^4}{32\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)} + \text{----------}\right]$$

$$\sqrt{\frac{\pi x}{2}}\ J_{\frac{3}{2}}(x) = \sqrt{\frac{\pi x}{2}}\ \frac{2x\sqrt{x}}{3\sqrt{2}\ \sqrt{\pi}}\left[1 - \frac{x^2}{4\left(\frac{5}{2}\right)} + \frac{x^4}{32\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)} + \text{----------}\right]$$

$$= \frac{x^2}{3}\left[1 - \frac{x^2}{2.5} + \frac{x^4}{5.7.8} + - - - - - - - - - - - \right]$$

$$\sqrt{\frac{\pi x}{2}}\ J_{\frac{3}{2}}(x) = \frac{x^2}{3} - \frac{x^4}{2.3.5} + \frac{x^6}{3.5.7.8} + \text{-----------}$$

$$= \frac{2x^2}{2.3} - \frac{4x^4}{2.3.4.5} + \frac{6x^6}{2.3.4.5.6.7} + \text{-----------}$$

$$= \frac{2x^2}{3\,!} - \frac{4x^4}{5\,!} + \frac{6x^6}{7\,!} + \text{-----------}$$

$$= \left(\frac{1}{2!} - \frac{1}{3!}\right)x^2 - \left(\frac{1}{4\,!} - \frac{1}{5\,!}\right)x^4 + \left(\frac{1}{6\,!} - \frac{1}{7\,!}\right)x^6 - + \text{-----------------}$$

$$= \left(\frac{x^2}{2!} - \frac{x^2}{3!}\right) - \left(\frac{x^4}{2!} - \frac{x^4}{5\,!}\right) + \left(\frac{x^6}{6\,!} - \frac{x^6}{7\,!}\right) - + \text{-----------------}$$

$$= \left[\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \text{------------}\right] + \left[-\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} - \text{------------}\right]$$

$$= -\left[-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \text{------------}\right] + \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} - \text{------------} - 1\right]$$

$$= -\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \text{------------} - 1\right] + \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \text{------------}\right] - 1$$

$$= -[\mathbf{Cos}\ x - 1] + \frac{1}{x}[\mathbf{Sin}\ x] - 1 = \frac{1}{x}\sin x - \cos x$$

Theorem: Prove that $\frac{d}{dx}\left[\frac{J_{-n}}{J_n}\right] = -\frac{2\sin n\pi}{\pi {J_n}^2}$

Proof: By the definition of Bessel's formula

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

But J_n *and* J_{-n} *are solutions of the Differential Equation*

$$\therefore \frac{d^2J_n}{dx^2} + \frac{1}{x}\frac{dJ_n}{dx}\left(1 - \frac{n^2}{x^2}\right)J_n = 0 \quad \text{-------(1)} \times J_{-n}$$

$$\text{And}\quad \frac{d^2J_{-n}}{dx^2} + \frac{1}{x}\frac{dJ_{-n}}{dx}\left(1 - \frac{n^2}{x^2}\right)J_{-n} = 0 \quad \text{------(2)} \times J_n$$

$$\left[J_{-n}\frac{d^2J_n}{dx^2} - J_n\frac{d^2J_{-n}}{dx^2}\right] - \frac{1}{x}\left[J_{-n}\frac{dJ_n}{dx} - J_n\frac{dJ_{-n}}{dx}\right] = 0$$

$$\frac{\left[J_{-n}\frac{d^2J_n}{dx^2}-J_n\frac{d^2J_{-n}}{dx^2}\right]}{\left[J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}\right]}=-\frac{1}{x}$$

Integrating on both sides $\int \frac{\left[J_{-n}\frac{d^2J_n}{dx^2}-J_n\frac{d^2J_{-n}}{dx^2}\right]}{\left[J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}\right]}\, dx = -\int \frac{1}{x}\, dx$

$$\log[\, J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx}] = -\log x + \log A = \log\frac{A}{x}$$

$$J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx} = \frac{A}{x} \text{----------(3)}$$

$$\Rightarrow \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\ \Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}\times\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\left(\frac{1}{2}\right)$$

$$-\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\ \Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\times\sum_{r=0}^{\infty}\frac{(-1)^r(-n+2r)}{r\,!\ \Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}\left(\frac{1}{2}\right)=\frac{A}{x}$$

Comparing the coefficient of $\frac{1}{x}$

$$\frac{\mathbf{1}}{\boldsymbol{\Gamma(-n+1)\Gamma(n+1)}}[\,\boldsymbol{n-(-n)}] = \boldsymbol{A}$$

$$\boldsymbol{A} = \frac{\mathbf{2}\boldsymbol{n}}{\boldsymbol{\Gamma(-n+1)\Gamma(n+1)}} = \frac{\mathbf{2}}{\boldsymbol{\Gamma(-n+1)\Gamma(n)}} = \frac{\mathbf{2}}{\boldsymbol{\Gamma(n)\Gamma(1-n)}} = \frac{\mathbf{2}}{\frac{\boldsymbol{\pi}}{\boldsymbol{\sin n\pi}}} = \frac{\mathbf{2}\,\boldsymbol{\sin n\pi}}{\boldsymbol{\pi}}$$

Put the value in (3)

$$J_{-n}\frac{dJ_n}{dx}-J_n\frac{dJ_{-n}}{dx} = \frac{A}{x} = \frac{\mathbf{2}\,\boldsymbol{\sin n\pi}}{\boldsymbol{\pi x}}$$

$$J_n\frac{dJ_{-n}}{dx}-J_{-n}\frac{dJ_n}{dx} = -\frac{\mathbf{2}\,\boldsymbol{\sin n\pi}}{\boldsymbol{\pi x}}$$

Dividing with $\boldsymbol{J^2{}_n}$ on both sides

$$\frac{J_n\frac{dJ_{-n}}{dx}-J_{-n}\frac{dJ_n}{dx}}{\boldsymbol{J^2{}_n}} = -\frac{\mathbf{2}\,\boldsymbol{\sin n\pi}}{\boldsymbol{\pi x J^2{}_n}}$$

$$\textbf{Hence } \frac{d}{dx}\left[\frac{J_{-n}}{J_n}\right] = -\frac{2\sin n\pi}{\pi {J_n}^2}$$

Theorem: Prove that

$$\frac{d}{dx}\left[{J_n}^2+J^2{}_{(n+1)}\right] = 2\left[\ \frac{n}{x}{J_n}^2-\frac{n+1}{x}J^2{}_{(n+1)}\right]$$

Proof: $\frac{d}{dx}\left[{J_n}^2+J^2{}_{(n+1)}\right] = 2J_nJ'_n+2J_{n+1}J'_{n+1}$ -----------(1)

By Recurrence Formula – I

$$xJ'_n(x) = n\ J_n(x) - x\,J_{n+1}(x)$$

$$\Rightarrow J'_n = \frac{n}{x} J_n - J_{n+1} \text{ -----------(2)}$$

By Recurrence Formula – II

$$xJ'_n(x) = -n\ J_n(x) + x\,J_{n-1}(x)$$

$$J'_n = -\frac{n}{x} J_n + J_{n-1}$$

$Replacing\ n\ by\ (n+1)$

$$J'_{n+1} = -\frac{n+1}{x} J_{n+1} + J_n \text{ ----------------(3)}$$

Put the values (2) & (3) in (1)

:

$$\frac{d}{dx}\left[J_n{}^2 + J^2{}_{(n+1)}\right] = 2J_nJ'_n + 2J_{n+1}J'_{n+1}$$

$$= 2J_n\left[\frac{n}{x} J_n - J_{n+1}\right] + 2J_{n+1}\left[-\frac{n+1}{x} J_{n+1} + J_n\right]$$

$$= \frac{2n}{x}J_n{}^2 - 2J_n J_{n+1} - \frac{2(n+1)}{x}J^2{}_{n+1} + 2J_n J_{n+1}$$

$$= 2\left[\frac{n}{x}J_n{}^2 - \frac{n+1}{x}J^2{}_{(n+1)}\right]$$

Theorem: Prove that $\frac{d}{dx}(x\,J_n\,J_{n+1}) = x[J_n{}^2 - J^2{}_{(n+1)}]$

Proof: We know that $(u\,v\,w)' = u'v\,w + u\,v'w + u\,v\,w'$

Now $\frac{d}{dx}(x\,J_n\,J_{n+1}) = J_n\,J_{n+1} + x\,J'_n\,J_{n+1} + x\,J_n\,J'_{n+1}$ ------(1)

By Recurrence Formula – I

$$xJ'_n = n\ J_n - x\,J_{n+1} \text{ ----------(2)}$$

By Recurrence Formula – II

$$xJ'_n(x) = -n\ J_n(x) + x\,J_{n-1}(x)$$

$Replacing\ n\ by\ (n+1)$

$$xJ'_{n+1} = -(n+1)\ J_{n+1} + x\,J_n \text{ ----------(3)}$$

Put the values (2) &(3) in (1)

$$\frac{d}{dx}(x\,J_n\,J_{n+1}) = J_n\,J_{n+1} + (x\,J'_n)\,J_{n+1} + (x\,J'_{n+1})\,J_n$$

$$= J_n\,J_{n+1} + (n\ J_n - x\,J_{n+1})\,J_{n+1} + (-(n+1)\ J_{n+1} + x\,J_n)\,J_n$$

$$= J_n\,J_{n+1} + n\ J_n\,J_{n+1} - x\,J^2{}_{(n+1)} - (n+1)\ J_n\,J_{n+1} + xJ_n{}^2$$

$$= x[J_n{}^2 - J^2{}_{(n+1)}]$$

&&&&&&&

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Special Functions VII A

2.Power series and power series solutions of Ordinary differential Equations

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Infinite series:

A series is the sum of the terms in the sequence $< u_n >$. that is

if $< u_n > = < u_1, u_2, u_3, \ldots, u_n, \ldots >$ the it's infinite sum

$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$

is called an Infinite series and is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

Example: Let $< \frac{1}{n} > = < \frac{1}{1}, \frac{1}{2}, \ldots \frac{1}{n}, \frac{1}{n+1}, \ldots >$ is a sequence of positive terms then

$\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}, + \frac{1}{n+1} \ldots = \sum \frac{1}{n}$ is an infinite series.

Cauchy's nth root test. Let $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \sqrt[n]{u_n} = l$

then i) If $l < 1$ convergent. ii) If $l > 1$ divergent iii) If $l = 1$ test fails.

D' Alembert's Ratio Test on convergence of a series. Let $\sum u_n$ is a series of positive terms such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

i)If $l < 1$ Series Convergent. ii) If $l > 1$ Series Divergent. iii) If $l = 1$ test fails.

Power series: An infinite series are in the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots$$

called power series

Radius of Convergent series:

A positive number r is said to be the radius of convergence of a power series if the power series convergent for every $|x| < r$ and divergent for every $|x| > r$

Theorem:

If the power series $\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots$ is such that $a_n \neq 0$ for all n and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{r}$$

then $\sum_{n=0}^{\infty} a_n\, x^n$ convergent for $|x| < r$ and divergent for every $|x| > r$

Proof: Let $u_n = a_n x^n$ for all n

$u_{n+1} = a_{n+1} x^{n+1}$ for all n

Now

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}\, x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x| = \frac{|x|}{r} \quad \text{--(1)} \quad (\because \textit{Given} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{r})$$

By D'Alembert's Ratio test

a. $\frac{|x|}{r} < 1 \Rightarrow |x| < r$ Convergent b. $\frac{|x|}{r} > 1 \Rightarrow |x| > r$ divergent

Theorem:

If the power series $\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots$ is such that $a_n \neq 0$ for all n and

$$\lim_{n\to\infty} a_n{}^{1/n} = \frac{1}{r}$$

then $\sum_{n=0}^{\infty} a_n\, x^n$ convergent for $|x| < r$ and divergent for every $|x| > r$

Proof: Let $u_n = a_n x^n$ for all n

$$\lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} |a_n x^n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n} |x| = \frac{|x|}{r}$$

By Cauchy's nth root test

a. $\frac{|x|}{r} < 1 \Rightarrow |x| < r$ Convergent b. $\frac{|x|}{r} > 1 \Rightarrow |x| > r$ divergent

Note: Also, you find the radius r of the convergence of the power series.

Problem – 1: Find the radius of the convergent series $\sum \frac{(n+1)x^n}{n(n+2)}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{(n+1)}{n(n+2)} \qquad a_{n+1} = \frac{(n+2)}{(n+1)(n+3)}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)(n+3)} \times \frac{n(n+2)}{(n+1)} = \lim_{n\to\infty} \frac{n(n+2)^2}{(n+1)^2(n+3)} = \lim_{n\to\infty} \frac{n^3(1+\frac{2}{n})^2}{n^3\left(1+\frac{1}{n}\right)^2(1+\frac{3}{n})} = 1$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1$$

2.Find the radius of the convergent series $\sum \frac{2^n x^n}{n\,!}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{2^n}{n\,!} \qquad a_{n+1} = \frac{2^{n+1}}{(n+1)\,!}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)\,!} \times \frac{n\,!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = \infty$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{\infty} = 0$$

3. Find the radius of the convergent series $\sum \frac{n^n x^n}{n\,!}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{n^n}{n\,!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)\,!}$$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\,!} \times \frac{n\,!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e}$$

4. Find the radius of the convergent series $\sum \frac{(-1)^n x^{2n}}{(n\,!)^2 2^{2n}}$

Solution: First to compare the given series with $\sum_{n=0}^{\infty} a_n\, x^n$

$$a_n = \frac{(-1)^n}{(n\,!)^2 2^{2n}} \qquad a_{n+1} = \frac{(-1)^{n+1}}{((n+1)\,!)^2 2^{2n+2}}$$

Now $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{((n+1)\,!)^2 2^{2n+2}} \times \frac{(n\,!)^2 2^{2n}}{1} = \lim_{n\to\infty} \frac{1}{4(n+1)^2} = 0$

$$\therefore Radius = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty$$

Power series solutions of Ordinary differential Equations

Problems: 1.Solve by power series method $y' - y = 0$

Solution: Let $y = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\ldots\ldots$

Put the values in the given differential equation $y' - y = 0$

$\Rightarrow [\, a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots\ldots\ldots\ldots]$

$- [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots\ldots\ldots\ldots = 0$

$\Rightarrow [\, (a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + (4a_4 - a_3)x^3 + (5a_5 - a_4)x^4 + \ldots\ldots\ldots = 0$

Comparing the corresponding coefficients on both sides

$a_1 - a_0 = 0$ ----(1) $\Rightarrow$ $a_1 = a_0$

$2a_2 - a_1 = 0$ ---(2) $\Rightarrow 2a_2 = a_1 = a_0 \Rightarrow a_2 = \frac{a_0}{2} = \frac{a_0}{2!}$

$3a_3 - a_2 = 0$ ---(3) $\Rightarrow 3a_3 = a_2 = \frac{a_0}{2} \Rightarrow a_3 = \frac{a_0}{6} = \frac{a_0}{3!}$

$4a_4 - a_3 = 0$ ---(4) $\Rightarrow 4a_4 = a_3 = \frac{a_0}{6} \Rightarrow a_4 = \frac{a_0}{24} = \frac{a_0}{4!}$ Etc put the values in

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

$y = a_0 + a_0x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_0}{5!}x^5 + \ldots\ldots\ldots\ldots$

$y = a_0[1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \ldots\ldots\ldots\ldots]$

2.Solve by power series method $y'' - 4y = 0$

Solution: Let $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \ldots\ldots\ldots\ldots$

And $y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots\ldots\ldots\ldots$

Put the values in the given differential equation $y'' - 4y = 0$

$\Rightarrow [2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \ldots\ldots\ldots\ldots]$

$- 4[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots = 0$

$\Rightarrow [(2a_2 - 4a_0) + (6a_3 - 4a_1)x + (12a_4 - 4a_2)x^2 + (20a_5 - 4a_3)x^3 + \ldots\ldots\ldots = 0$

Comparing the corresponding coefficients on both sides

$2a_2 - 4a_0 = 0$ ----(1) $\Rightarrow$ $a_2 = 2a_0$

$6a_3 - 4a_1 = 0$ ---(2) $\Rightarrow 6a_3 = 4a_1 \Rightarrow a_3 = \left(\frac{2}{3}\right)a_1 = \frac{4}{3}a_1$

$12a_4 - 4a_2 = 0$ ---(3) $\Rightarrow 12a_4 = 4a_2 \Rightarrow a_4 = \frac{1}{3}a_2 = \frac{2}{3}a_0$

$20a_5 - 4a_3 = 0$ ---(4) $\Rightarrow 20a_5 = 4a_3 \Rightarrow a_5 = \frac{1}{5}a_3 = \frac{4}{15}a_1$

Put the values in

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$

$y = a_0 + a_1x + 2a_0x^2 + \frac{4}{3}a_1x^3 + \frac{2}{3}a_0x^4 + \frac{4}{15}a_1x^5 + \ldots\ldots\ldots\ldots$

$y = a_0[1 + 2x^2 + \frac{2}{3}x^4 + \ldots\ldots\ldots\ldots] + a_1[x + \frac{4}{3}x^3 + \frac{4}{15}x^5 + \ldots\ldots\ldots]$

3.Find the power series solution of the equation $(x^2-1)y''+xy'-y=0$

Solution: Let $y=\sum_{n=0}^{\infty} a_n\, x^n = a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ldots\ldots\ldots\ldots$

And $y'' = 2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots\ldots\ldots\ldots$

Put the values in the given differential equation

$$(x^2-1)y''+xy'-y=0$$

$$(x^2-1)(2a_2+6a_3x+12a_4x^2+20a_5x^3+\ \ldots\ldots\ldots\ldots)$$
$$+x(a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ \ldots\ldots\ldots\ldots)$$
$$-(a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ \ldots\ldots\ldots\ldots)=0$$

Comparing the corresponding coefficients

Consistent $-2a_2-a_0=0\ \Rightarrow\ a_2=-\frac{a_0}{2}$

x- coefficient $-6a_3+a_1-a_1=0 \Rightarrow\ a_3=0$

x^2- coefficient $2a_2-12a_4+2a_2-a_2=0$

$$\Rightarrow\ -12a_4+3a_2=0 \Rightarrow\ -12a_4=\ -3a_2=-3\left(-\frac{a_0}{2}\right)=\frac{3a_0}{2} \Rightarrow\ a_4=-\frac{a_0}{8}$$

x^3- coefficient $6a_3-20a_5+3a_3-a_3=0 \Rightarrow$ But $a_3=0$ hence $a_5=0$

Put the values in

$$y=a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$$

$$y=a_0+a_1x+\left(-\frac{a_0}{2}\right)x^2+(0)x^3+\left(-\frac{a_0}{8}\right)x^4+(0)x^5+\ldots\ldots\ldots\ldots$$

$$\boldsymbol{y=a_0[1-\frac{x^2}{2}+\frac{x^4}{8}+\ldots\ldots\ldots\ldots]+a_1x}$$

4.Find the power series solution of the equation $(x^2+1)y''+xy'-xy=0$

Solution: Let $y=\sum_{n=0}^{\infty} a_n\, x^n = a_0+a_1x+a_2x^2+a_3x^3+\ a_4x^4+a_5x^5+\ldots\ldots\ldots\ldots$

is the power series solution of the given differential equation

Now $y' = a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ldots\ldots\ldots\ldots$

And $y'' = 2a_2+6a_3x+12a_4x^2+20a_5x^3+\ldots\ldots\ldots\ldots$

Put the values in the given differential equation

$$(x^2+1)y''+xy'-xy=0$$

$$(x^2+1)(2a_2+6a_3x+12a_4x^2+20a_5x^3+\ \ldots\ldots\ldots\ldots)$$
$$+x(a_1+2a_2x+3a_3x^2+4a_4x^3+\ 5a_5x^4+\ \ldots\ldots\ldots\ldots)$$

$$-x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots) = 0$$

Comparing the corresponding coefficients

Consistent $2a_2 = 0 \Rightarrow a_2 = 0$

x- coefficient $6a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{1}{6}(a_0 - a_1)$

x^2- coefficient $2a_2 + 12a_4 + 2a_2 - a_1 = 0$

$$\Rightarrow 12a_4 + 4(0)a_2 - a_1 = 0 \Rightarrow 12a_4 = a_1 \Rightarrow a_4 = \frac{a_1}{12}$$

x^3- coefficient $20a_5 + 6a_3 + 3a_3 - a_2 = 0 \Rightarrow$ But $a_2 = 0$

$$20a_5 + 9a_3 = 0 \Rightarrow 20a_5 = -9a_3 = -9\left[\frac{1}{6}(a_0 - a_1)\right] = 0$$

$$\Rightarrow a_5 = -\frac{3}{40}(a_0 - a_1)$$

Put the values in

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots\ldots$$

$$y = a_0 + a_1x + (0)x^2 + \left[\frac{1}{6}(a_0 - a_1)\right]x^3 + \left(\frac{a_1}{12}\right)x^4 + \left[-\frac{3}{40}(a_0 - a_1)\right]x^5 + \ldots\ldots\ldots\ldots$$

$$\boldsymbol{y = a_0[1 + \frac{x^3}{6} - \frac{3x^5}{40} + \ldots\ldots\ldots\ldots] + a_1[x - \frac{x^3}{6} + \frac{x^4}{12} + \ldots\ldots\ldots\ldots]}$$

HW

5.Find the power series solution of the equation $y'' + xy' + x^2y = 0$

Ans: $\boldsymbol{y = a_0[1 - \frac{x^4}{12} + \frac{x^6}{90} + \ldots\ldots\ldots\ldots] + a_1[x - \frac{x^3}{6} - \frac{x^5}{40} + \ldots\ldots\ldots\ldots]}$

6.Find the power series solution of the equation $y'' - xy' + x^2y = 0$

Ans: $\boldsymbol{y = a_0[1 - \frac{x^4}{12} - \frac{x^6}{90} - \ldots\ldots\ldots\ldots] + a_1[x + \frac{x^3}{6} - \frac{x^5}{40} + \ldots\ldots\ldots\ldots]}$

All the best